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Problems in Taxation

An Optimization Approach for Loss Offset Options

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Abstract

We solve an optimization problem which arises in the German tax system. Here losses in some period can be tranferred to other periods reducing tax in these periods. Two variants of taxation can be applied. We formulate the problem as a mixed binary mathematical program and solve it via branch and bound using binary search. Special cases of the problem can be solved by fast polynomial algorithms.

1 Introduction

Decisions in business taxation do not exclusively focus on simple interpretation of tax law. In contrast, tax payers increasingly face complex decision problems in order to carry out optimal decisions for tax purposes. In this paper we focus on tax losses as one specific area of optimization problems that arises in most tax systems world wide. In detail we choose a problem that deals with Germany's income tax loss offset restrictions for individuals. Here losses in some period can be transferred to the previous assessment period or future assessment periods reducing tax in these periods.

Problems with options concerning loss offset restrictions in terms of optimization do not appear separately. In fact, tax systems provide lots of options that usually interact. The drawback is that all options have to be optimized simultaneously to achieve a global optimum. To meet that problem we expand the basic problem of loss offset optimization and implement a further option in our problem. The respective option deals with the assessment or flat rate taxation of capital income.

The remainder of this paper is organized as follows: Chapter 2 deals with the related literature of prior research in tax optimization problems as well as the optimization technique. In Chapter 3 we present the problem formulation that is transferred into the mathematical programming formulation in Chapter 4. An

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approach to solve the formulated program is discussed in Chapter 5 and an example problem in Chapter 6 illustrates the developed algorithm. Chapter 7 concludes.

2 Related Work

Prior research of tax related optimization problems is quite managable. The vast majority of tax related literature of optimization problems concentrates on problems from a juridical point of view or from a macroeconomic perspective. The distinctive juridical branch of literature is based on verbal description of problems in taxation. Indeed individual situations are analyzed. However, usually the decision setting is restricted to a static situation.

Economists focus on formal optimization or empirical investigations in taxation. Basically, dynamic models are used. However, only from a macroeconomic perspective e.g. in terms of implementing tax systems that cause minimal distortions[1] or cross border investigations that mainly derive optimal repatriation strategies[2]. The individual decision setting is completely neglected. Another focus of research in the field of economics are distortions caused by asymetric taxation of debt and equity.

Business taxation approaches optimization from an individual point of view. Literature in this field is quite rare. Niemann (2006), Knirsch/Schanz (2008) and Schanz (2008) investigate optimal repatriation amounts by applying a business tax planning model. The results are based on heuristic approaches as tabu search or scatter search.

To our knowledge there are no existing contributions that deal with technical optimization of problems on the individual tax payers level except the contributions mentioned above.

3 Problem

Given: In each period $t = 1, 2, \ldots, T$ a tax payer earns exogenous cash flows and endogenous capital yield. Cash inflow (less depreciation) and capital yield for all periods are known in advance. Tax payments must be made on cash inflows and capital yield in each period according to one or more given tax functions. In case of a negative cash inflow (loss) in some period t this can either be carried

[1] E.g. see Sandmo (1976). International tax planning matters are discussed by Alworth (1988).

[2] Eg. see Altshuler/Newlon/Randolph/William (1995), Desai/Foley/Hines (2001) and Altshuler/Grubert (2003).

backward to period $t-1$, which results in a tax refund in t, or carried forward to periods $t+1,\dots,T$, where future taxable income in $t+1,\dots,T$ will be reduced.

Starting with some initial wealth W_0 the terminal wealth W_T at the end of the last period is the result of cash inflow plus capital yield minus tax payments in all prior periods. In each period one can choose between two taxation variants:

(i) capital yield and cash inflow are added and taxed according to a tax function;

(ii) capital yield is taxed at a flat rate of $r = 0,25$; cash inflow is taxed according to a tax function.

Question: Which taxation variant should be chosen in each period and which amount of losses should be carried back in order to maximize terminal wealth at the end of period T?

For periods t we define:

(1) A_t: cash inflow in period t

(2) B_t: capital income in period t

(3) I_t: taxable income in period t

(4.1) Tax function defined in Sec. 32a of the German income tax code (EStG)

$$f(I) = \begin{cases} 0 & \text{for} \quad 0 \le I < 8,004 \\ (912.17 \cdot a + 1,400) \cdot a & \text{for} \quad 8,005 \le I \le 13,469 \\ (228.74 \cdot b + 2,397) \cdot b + 1,038 & \text{for} \quad 13,470 \le I \le 52,881 \\ 0.42 \cdot I - 8,172 & \text{for} \quad 52,882 \le I \le 250,730 \\ 0.45 \cdot I - 15,694 & \text{for} \quad I \ge 250,731 \end{cases}$$

with $a = \frac{(I-8{,}004)}{10{,}000}$ and $b = \frac{(I-13{,}469)}{10{,}000}$

(4.2) Tax payment:

$g(B_t)$: $0,25 \cdot B_t$

$f(A_t + B_t)$: tax payment on cash inflow and capital income calculated using tax function $f(.)$

$f(A_t)$: tax payment on cash inflow calculated using tax function $f(.)$

$g(B_t)$: tax payment on capital yield using 25% flat rate

(5) $W_t = f(S1_t, S2_t)$: terminal wealth in period t
where

(6.1) $S1_t = f(A_t + B_t)$

(6.2) $S2_t = f(A_t) + g(B_t)$

W.l.o.g. we assume that $B_t = i \cdot W_{t-1}$ where i is a constant interest rate for $t = 1, \ldots, T$

Here we concentrate on a $T = 3$ period problem with a single loss period; we assume w.l.o.g.

(a) loss period is $t = 2$ such that loss carry backward to $t = 1$ and loss carry forward to $t = 3$ is possible;

(b) the value of depreciations is zero in each period

(c) cash inflows in $t = 1$ and $t = 3$ can take any positive value

In practical settings the problem is solved sequentially in two steps:

(1) calculate tax to be paid for $t = 1$

(2) calculate possible tax payments for $t \geq 1$ after considering loss shifting and receive possible tax refund at the beginning of $t = 2$.

The two step procedure is due to the fact that losses from $t = 2$ and profits from $t = 3$ are not known when calculating tax for $t = 1$.

Here we solve the problem assuming that all necessary data are known. We calculate the tax for all periods $t = 1, 2, 3$ simultaneously.

Lemma: *The terminal wealth of the simultaneous approach is never smaller than this of the sequential one.*

Proof: Terminal wealth for period 3 is calculated by

$$
\begin{aligned}
W_3 &= W_0 + \sum_{t=1}^{3} (A_t + B_t) - \sum_{t=1}^{3} S_t \\
&= W_0 + \sum_{t=1}^{3} A_t + \sum_{t=1}^{3} (B_t - S_t)
\end{aligned}
$$

The starting wealth W_0 and cash inflows A_t, $t = 1, \ldots, 3$ are constants having the same value for both approaches. B_t and S_t, $t = 1, \ldots, 3$ depend on $x(2,1)$ (respectively on $x(2,3)$). In the simultaneous approach the maximum amount for $\sum_{t=1}^{3}(B_t - S_t)$ is found which maximizes terminal wealth. Thus the sequential approach cannot find a higher terminal wealth than the simultaneous approach.

4 Mathematical Programming Formulation

In this section we formulate a mathematical programm for the model according to the principles of Schmidt (1999).

The following MP (1) – (3) describes the problem:

$$\max \left(W_3 = W_1 + \sum_{t=1}^{3} AB_t - \sum_{t=1}^{3} S_t \right)$$

can be shortend to

$$\max \left(\sum_{t=1}^{3} (B_t - S_t) \right)$$

due to the Lemma in Section 3
s.t.

$$y_t \in \text{ from } \{0,1\} \tag{2.1}$$

$$x(t,t-1) \geq \max\{C;(1-y_t)\cdot(AB_t + y_t \cdot A_t)\} \tag{2.2}$$

$$x(t,t-1) \leq 0 \tag{3}$$

where

$$S_t = (1-y_t)\cdot 0 + y_t \cdot (0.25 \cdot B_t + 0) \tag{4.1}$$

$$\begin{aligned} &= (1-y_t)\cdot((912.17 \cdot a_t + 1,400)\cdot a_t) \\ &\quad + y_t \cdot (0.25 \cdot B_t + ((912.17 \cdot a_t + 1,400)\cdot a_t)) \end{aligned} \tag{4.2}$$

$$\begin{aligned} &= (1-y_t)\cdot((228.74 \cdot b_t + 2,397)\cdot b_t + 1,038) \\ &\quad + y_t \cdot (0.25 \cdot B_t + ((228.74 \cdot b_t + 2,397)\cdot b_t + 1,038)) \end{aligned} \tag{4.3}$$

$$= (1-y_t)\cdot(0.42 \cdot I_t - 8,172) + y_t \cdot (0.25 \cdot B_t + (0.42 \cdot I_t - 8,172)) \tag{4.4}$$

$$= (1-y_t)\cdot(0.45 \cdot I_t - 15,694) + y_t \cdot (0.25 \cdot B_t + (0.45 \cdot I_t - 15,694) \tag{4.5}$$

$$a_t = \frac{(I_t - 8,004)}{10,000} \tag{4.6}$$

$$b_t = \frac{(I_t - 13,469)}{10,000} \tag{4.7}$$

and for $t = 1$

$$
\begin{aligned}
I_t =& (1-y_t)\cdot \max\{AB_t + x(t+1,t);0\} + y_t \cdot \max\{A_t + x(t+1,t);0\} && (5.1)\\
I_{t+1} =& 0 && (5.2)\\
I_{t+2} =& (1-y_{(t+2)})\cdot \max\{AB_{(t+2)} + (1-y_{(t+1)})\cdot(AB_{(t+1)} - x(t+1,t)) && (5.3)\\
& + y_{(t+1)}\cdot(A_{(t+1)} - x(t+1,t));0\}\\
& + y_{(t+2)}\cdot \max\{A_{(t+2)} + (1-y_{(t+1)})\cdot(AB_{(t+1)} - x(t+1,t))\\
& + y_{(t+1)}\cdot(A_{(t+1)} - x(t+1,t));0\}
\end{aligned}
$$

where $AB_t = A_t + B_t$, C is some upper bound on possible loss carry backward, y_t is the taxation variant applied in period t and $x(t,t-1)$ is the loss carry-backward from period t to period $t-1$.

5 Solving the Mathematical Program

If either taxation variant (i) or taxation variant (ii) is allowed for all periods it is optimal to balance $x(t,t-1)$ and $x(t,t+1)$ depending on interest rate i and cash inflow A_{t-1} and A_{t+1}.

Theorem: For optimal balancing of $x(t,t-1)$ and $x(t,t+1)$ in case of either taxation variant (i) or (ii) we can use binary search to find the optimal solution.

Proof: Let $L <_{\min} \leq x(t,t-1) \leq L <_{\max}$ with $L <_{\min}$ and $L <_{\max}$ be the minimum and the maximum possible loss carry backward amount from period t to period $t-1$. Less $x(t,t-1)$ increases tax payment in $t-1$ but decreases tax payment in $t+1$; there is some optimal tradeoff between increase and decrease which can be found by binary search for $x_{(t,t-1)}$ in the interval $[L <_{\min}, L < \max]$. $L <_{\min}$ decreases $x(t,t-1)$ to a minimum and $L <_{\max}$ increases $x(t,t-1)$ to a maximum. Since $W(x)$ is a continuous function with a single increasing slope (reaching a maximum) and a single decreasing slope as shown in Figure 1.

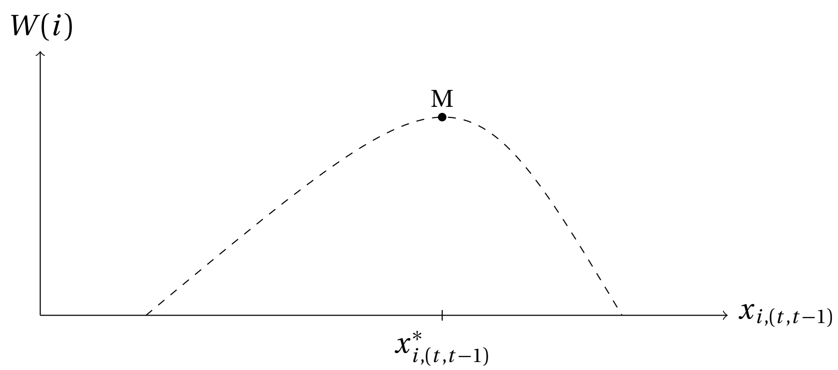

Fig. 1: Terminal wealth as a function of interest rate i

Now we investigate the problem where both taxation variants (i) and (ii) are possible in each period. Variant (ii) might help in two ways:

(1) in a loss period variant (ii) increases the amount of transferable loss at the cost of 25%; if additional tax reduction by loss transfer is greater than this cost it increases cost savings;

(2) in a non-loss period variant (ii) increases tax savings if the average tax is > 25%

The way to solve the general problem is to apply branch and bound to decide on the binary decision variables representing the taxation variants and to apply for each node binary search to find the optimal values of the real valued variable. A general description of the branch and bound algorithm can be found in the contribution of Lawler/Wood (1966). We calculate upper bounds for each node by relaxing the binary variables y_t to real valued variables. Thus the root node represents a real valued MP formulation. On the first level one binary decision variable is assumed and two are relaxed to real valued, on the second level two binary decision variables are assumed and one is relaxed to be real valued; on the third level all binary decision variables are fixed. For the $T = 3$ period problem we have in the worst case six leaves (111), (112), (122), (222), (221), (211) of the branching tree representing all possible decisions for combinations of taxation variants related to periods 1, 2, and 3.

6 Example Problem

In this section we provide a numerical example based for the $T = 3$ problem to demonstrate the search process.

Let the interest rate $i = 0,03$; the initial wealth W_0 and cash flows A_t be

$$\begin{aligned} W_0 &= 593,194 \\ A_1 &= 20,000 \\ A_2 &= -50,000 \\ A_3 &= 50,000 \end{aligned}$$

The application of the branch and bound algorithm is shown in Figure 2.

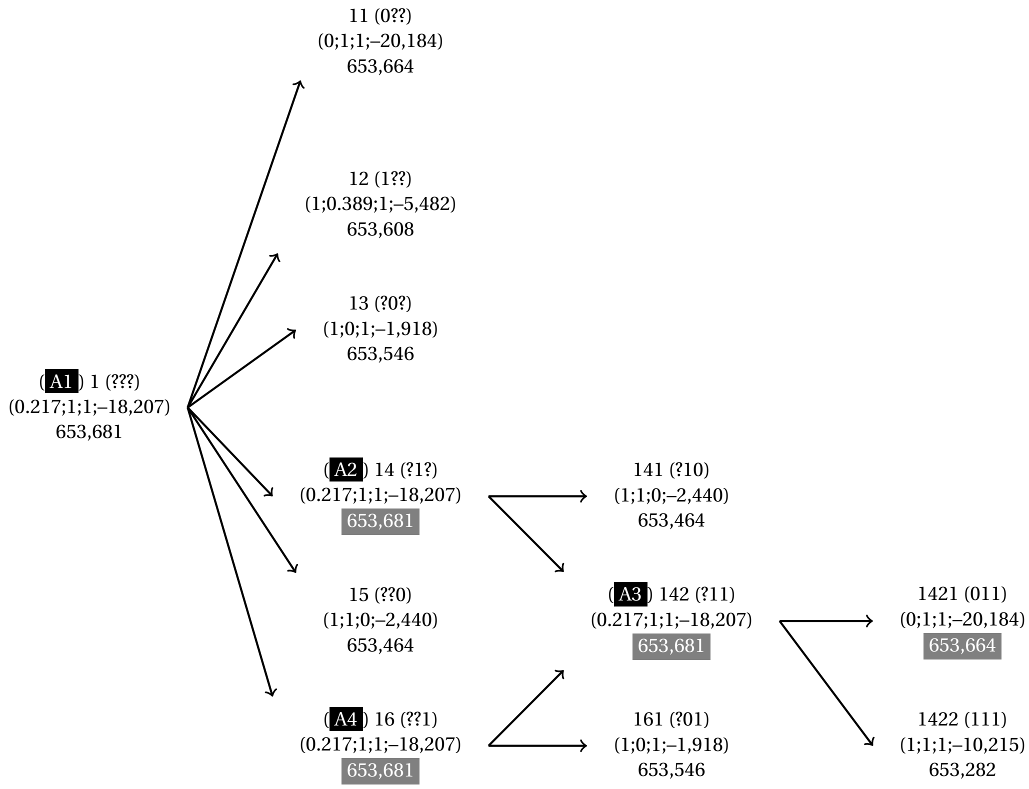

Fig. 2: B&B for example problem

Table 1 illustrates the example problem showing the root solution of the branch and bound tree.

t	0	1	2	3
A_t		20,000	–50,000	50,000
B_t		17,796	18,853	17,777
y_t		0.217	1	1
AB_t		37,796	–31,147	67,777
$x(t,t-1)$		0	–18,207	0
I_t		15,727	0	18,207
a_t		0.772	–0.800	1.020
b_t		0.226	–1.347	0.474
S_t		2,556	4,713	6,669
W_t	593,194	628,433	592,573	653,681

Tab. 1: Example problem and root solution

t	0	1	2	3
A_t		20,000	–50,000	50,000
B_t		17,796	18,868	17,792
y_t		0	1	1
AB_t		37,796	–31,132	67,792
$x(t, t-1)$		0	–20,184	0
I_t		17,612	0	20,184
a_t		0.961	–0.800	1.218
b_t		0.414	–1.347	0.672
S_t		2,070	4,717	7,199
W_t	593,194	628,920	593,070	653,664

Tab. 2: Example problem with optimal solution

Table 2 shows the optimal solution with $x(2,1) = -20.184$ generated by the branch and bound algorithm shown in Figure 2.

7 Conclusions

General problem with arbitrary number of periods can be solved along the same line. The number of binary variables y increases to T and the number of real variables x increases to the number of loss periods. Moreover the objective function has to be adopted to T periods.

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