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Abstract

It is well-known that stock prices fluctuate far more than dividends. Traditional valuation methods are not able to depict this fact. In this paper we incorporate excess volatility into a simple DCF model by considering an autoregressive cash flows process with random coefficients. We show that the model is free of arbitrage and that the transversality condition is met and we prove a valuation equation that differs from the classical Gordon-Shapiro version: Cost of capital (respectively dividend-price ratio) is stochastic and our model represents excess volatility. We discuss whether our assumptions are compatible with an equilibrium.

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1 Introduction

It is well-known that stock prices fluctuate far more than dividends over time. Shiller (1981) was first to demonstrate that stock prices exhibit *excess volatility* when compared to the discounted stream of ex post realized dividends. In his Nobel prize lecture he said in 2013:

> "It is hardly plausible that speculative prices make effective use of all information about probabilities of future dividends. It is far more plausible that the aggregate stock market price changes reflect inconstant perceptions, changes which Keynes referred to with the term 'animal spirits,' changes that infect the thinking even of the most of the so-called 'smart money' in the market".[1]

Cochrane (2011) emphasized that this phenomenon is not restricted to shares: Rather, it is present for all asset classes such as treasuries, bonds, foreign exchange, houses.[2] However, a fact like this cannot be easily explained by classical valuation models.

We want to engage in the discussion with the simple model of stochastic discounted cash flow (DCF) from Kruschwitz and Löffler (2019). This model is not focused on a particular asset class and furthermore incorporates the stochastic nature of cash flows and values. For the sake of convenience, we will always speak of enterprise or firm values in the following, despite the general context.

We limit our comments to a discrete-time model. Dividends and interest are usually not paid continuously, and this approach allows for arbitrary distributions for the dividend-price ratio. For the risk-free interest rate, however, we have used the continuous-time notation because we believe it is more readable.

Let us start with a meaningful interpretation of our concept. In the following we will speak of excessive volatility if the *future coefficient of dispersion* (the quotient of standard deviation and expected value) of the firm value is greater than the coefficient of dispersion of the company's cash flows.[3] It may well be possible to formulate excess volatility with a different measure or other statistical properties, but at present we want to commit ourselves as stated.

[1] See Shiller (2014, p. 1497).
[2] See Cochrane (2011, section I.C).
[3] Cf. Leroy and Porter (1981) who used the same measure.

Let us motivate our interpretation. If we assume that the cash flows CF_t of a financial asset form a martingale and, second, that the (deterministic) cost of capital k is constant over time, then[4]

$$V_t = \frac{CF_t}{k} \tag{1}$$

applies, V_t being the stock price of the asset. It follows directly from (1) that the coefficient of dispersion for cash flows is identical to the corresponding term for values because k and g are assumed to be deterministic. As we have seen it is exactly this property that runs contrary to the empirical fact mentioned in the beginning.

Seen in this light we are investigating whether an inequality of the form

$$\frac{\sigma[V_t]}{\mathrm{E}[V_t]} > \frac{\sigma[CF_t]}{\mathrm{E}[CF_t]}. \tag{2}$$

can be established within the DCF theory. In order to do so we will make two contributions.

1. We show that the traditional approach is based on an autoregressive cash flow process with deterministic coefficients under the risk-neutral probability measure Q. We prove that the use of precisely this property cannot allow for excessive volatility. Rather, it can be shown that the two coefficients of dispersion will inevitably coincide.

2. We expand the DCF model and consider an autoregressive process for cash flows with random coefficients. We are able to show that, first, this model is free of arbitrage and, second, the transversality condition is met. However, the established pricing equation differs considerably from (1): Now the cost of capital (more precisely the dividend-price ratio) will be a random variable κ_t which is independent of cash flows. Put differently, the valuation equation reads

$$V_t = \frac{CF_t}{\kappa_t}. \tag{3}$$

   It is immediate that now excess volatility can be explained, (2) will certainly hold.

It must be left to future research whether our idea can be substantiated by empirical examinations.

[4] The equation is usually named after Gordon and Shapiro (1956), but can already be found (ignoring uncertainty) in Williams (1938, p. 56) and even Wiese (1930, p. 5).

2 A Theory of Stochastic Cost of Capital

2.1 Assumptions

We start with the assumptions of the model and at this stage we focus only on arbitrage. First, there is an risk-free short rate e^r, which for simplicity is constant over time. Second, we need a technical assumption that enables us to change expectation and limes:[5]

Cash flows: In the case of positive short rates $r > 0$ we assume that a cash flow process $(CF_t)_{t=1,\ldots}$ exists and we only presuppose that the cash flows have a lower bound K.

If short rates are zero or negative, cash flows have to be non-negative.

We do not assume that the lower bound K is zero; it may be arbitrarily small (negative). We only postulate that K is independent of time t as well as the state of nature.

Furthermore, there exists another process $(V_t)_{t=1,\ldots}$, which we shall call the "associated pricing process". In order for this designation to be meaningful and, in particular, for the pricing process to be unambiguous, several things must be taken into account.

Fundamental theorem: We assume that there is a risk neutral probability measure Q.[6]

For the pricing process the so-called fundamental theorem of price theory must apply. This theorem states that the return of holding the asset is riskless under the risk neutral probability measure,

$$\mathrm{E}_Q[V_{t+1} + CF_{t+1}|\mathcal{F}_t] = e^r \cdot V_t. \tag{Fund}$$

Transversality: The uniqueness of the pricing process is usually ensured by a transversality condition. If bubble solutions as, for example, in Froot and Obstfeld (1991, p. 1192), are ruled out firm values will always be unique.

[5] Compare Kruschwitz and Löffler (2013, assumption 1) why this assumption is necessary.

[6] The existence of such a measure can be established if the market is free of arbitrage as it was first shown (in a general setup) by Harrison and Kreps (1979).

In the literature there is often a formulation that refers to the cost of capital k:

$$\forall t \qquad \lim_{T\to\infty} (1+k)^{t-T}\ \mathrm{E}[V_T|\mathcal{F}_t] = 0.$$

We consider this formulation to be inappropriate in the context to be discussed here because it necessarily presupposes that the cost of capital is deterministic. Assuming this, it inevitably follows that the coefficients of dispersion of cash flows and enterprise values are identical. For this reason we will not go along with this formulation.

Instead, we propose a formulation that uses the risk-neutral measure Q, i.e.,

$$\forall t \qquad e^{(t-T)r}\ \mathrm{E}_Q[V_T|\mathcal{F}_t] \xrightarrow{a.s.} 0, \tag{Trans}$$

where the convergence is Q-a.e., i.e., the set of all states that converge for $T\to\infty$ has full measure. This formulation has the advantage of not making any assumptions about the cost of capital.

Together, the conditions (Fund) and (Trans) ensure that the pricing process V_t is unique. The valuation will be given by

$$\forall t \qquad \sum_{s=t+1}^{T} e^{(t-s)r}\ \mathrm{E}_Q[CF_s|\mathcal{F}_t] \xrightarrow{a.s.} V_t. \tag{Value}$$

Again, the convergence is Q-a.e. for $T\to\infty$.

2.2 The Classical Approach

In order to generate an important interim result, we will initially concentrate on the classical DCF model with deterministic cost of capital. For simplicity, we focus on the case where the cost of capital k is constant over time.

Furthermore, cash flows are assumed to form an autoregressive process with deterministic coefficients using the subjective probability or[7]

$$\mathrm{E}[CF_{t+1}|\mathcal{F}_t] = e^g \cdot CF_t. \tag{4}$$

The growth rate g may be negative; cost of capital $1+k$ must be larger than e^g. From Kruschwitz and Löffler (2019, theorem 2.3) we get

$$\frac{\mathrm{E}_Q[CF_s|\mathcal{F}_t]}{e^{(s-t)r}} = \frac{\mathrm{E}[CF_s|\mathcal{F}_t]}{(1+k)^{s-t}} = \frac{e^{(s-t)g}\,CF_t}{(1+k)^{s-t}}$$

[7] This condition has a long history. It was (implicitly) used already in Miles and Ezzell (1980) and can be found in Leroy and Porter (1981, equation (2)) as well.

and therefore

$$\mathrm{E}_Q[CF_{t+1}|\mathcal{F}_t] = \frac{e^{r+g}}{1+k} CF_t. \tag{5}$$

Thus, we find that the cash flows CF also form an autoregressive process with deterministic coefficients *using the risk-neutral measure* Q. This is a rather surprising result. Additionally, the growth factor satisfies

$$e^{g'} := \frac{e^{r+g}}{1+k} < e^r$$

and this even applies to short rates r other than positive.

Condition (5) is generally unsuitable for modelling excessive volatility. In fact, if we assume that the cash flows meet such an AR(1) condition, then the coefficient of dispersion of cash flow and of firm value are necessarily identical as the following calculation shows:

$$\begin{aligned} V_t &= \sum_{s=t+1}^{\infty} e^{(t-s)r} \, \mathrm{E}_Q[CF_s|\mathcal{F}_t] \\ &= \sum_{s=t+1}^{\infty} e^{(s-t)(g'-r)} \, CF_t = \frac{CF_t}{e^{r-g'}-1}. \end{aligned} \tag{6}$$

The above assertion results from the fact that $\frac{1}{e^{r-g'}-1}$ is not random.

Replacing constant discount rates k with time-varying but deterministic cost of capital k_t will not change the relation between the coefficients of dispersion for cash flows and for firm values; the coefficients are still coinciding.[8] This should not come as a surprise since already Shiller remarked "…that the movements in expected real interest rates that would justify the variability in stock prices are very large – much larger than the movements in nominal interest rates over the sample period".[9]

Based on the above considerations, it is clear that in order to model excess volatility, we need an approach in which the cash flows do not follow a deterministic AR(1) process under Q. It might be believed that the desired result can be achieved with a higher order of the autoregressive process. But this simply is not the case. In the appendix (section 3) we show with an example that in case of an AR(2) process under Q a situation can occur where the coefficient of dispersion of the firm value is even smaller than the

[8]For a formal proof with time-dependent and deterministic cost of capital within our model see Laitenberger and Löffler (2006, proposition 1).

[9]See Shiller (1981, p. 434).

coefficient of dispersion of the corresponding cash flows.[10] However, this is most definitely contrary to what is observed empirically. Therefore, we strongly believe that a solution of the excess volatility problem must be found by using a completely new approach. The new concept consists in abandoning the idea of deterministic cost of capital.

2.3 Autoregressive Cash Flows with Random Coefficients

We consider all the assumptions described in section 2.1 to be reasonable and necessary. In context of *deterministic* cost of capital, they furthermore turned out to be appropriate. If, however, one considers *stochastic* cost of capital, this usefulness vanishes. In any case, despite intensive efforts, we have not been able to discover a way to come up with an appropriate derivation. For this reason, we do not consider the assumptions (Fund) and (Trans) to be expedient and will now present a different approach.

To this end we start with the transversality condition. Instead of looking at the final value V_T and its properties we concentrate on the long tail of the sum of cash flows instead, i.e., $\sum_{s>t} e^{(t-s)r} \operatorname{E}_Q[CF_s|\mathcal{F}_t]$. We will use the so-called Cauchy criterion which then directly ensures convergence. Transversality is given iff for $T_1 \to \infty$

$$\sup_{T_2 \geq T_1} \left| \sum_{s=T_1}^{T_2} e^{(t-s)r} \operatorname{E}_Q[CF_s|\mathcal{F}_t] \right| \xrightarrow{a.s.} 0. \tag{Cauchy}$$

Then, the convergence of the sum is guaranteed.

Using assumption (Cauchy) proves to be highly useful. Now one only needs to *define* the value of the company via the valuation equation (Value). Together with the assumption (Cauchy) this definition guarantees that then also (Trans) and (Fund) are met. This is because the following can be shown:

Proposition 1. *The following conditions are equivalent*

$$\text{(Trans) \& (Fund)} \quad \Longleftrightarrow \quad \text{(Cauchy) \& (Value)}$$

Proof. We have to prove two statements and we start by assuming that (Trans) and (Fund) are given. From (Fund) by induction it follows

$$V_t = \sum_{s=t+1}^{T} e^{(t-s)r} \operatorname{E}_Q[CF_s|\mathcal{F}_t] + e^{(t-T)r} \operatorname{E}_Q[V_T|\mathcal{F}_t]. \tag{7}$$

[10] A similar argument was already made by Leroy and Porter (1981, Theorem 1), although they considered the case of a stationary cash flow process ($g < 0$ in our notation).

Because V_t is finite, (Trans) implies Q-a.e. for $T \to \infty$

$$\forall t \quad \sum_{s=t+1}^{T} e^{(t-s)r} \operatorname{E}_Q[CF_s|\mathcal{F}_t] \xrightarrow{a.s.} 0.$$

Hence, (Value) holds and since the sequence converges (Cauchy) must be satisfied.

Assume now that (Cauchy) holds true and the value is given by (Value). In order to prove the fundamental theorem we have to show (lim denotes the a.s.-limes)

$$\operatorname{E}_Q\left[CF_{t+1} + \lim_{T\to\infty} \sum_{s=1}^{T} e^{-sr} \operatorname{E}_Q[CF_{t+1+s}|\mathcal{F}_{t+1}]|\mathcal{F}_t\right] = \\ = e^r \lim_{T\to\infty} \sum_{s=1}^{T} e^{-sr} \operatorname{E}_Q[CF_{t+s}|\mathcal{F}_t]$$

which is equivalent to

$$\operatorname{E}_Q\left[\lim_{T\to\infty} \sum_{s=1}^{T} e^{-(s+1)r} \operatorname{E}_Q[CF_{t+1+s}|\mathcal{F}_{t+1}]|\mathcal{F}_t\right] = \lim_{T\to\infty} \operatorname{E}_Q\left[\sum_{s=1}^{T} e^{-sr} CF_{t+s}|\mathcal{F}_t\right].$$

As one can see the claim is shown if on the left hand side the a.s.-limes $\lim_{T\to\infty}$ and expectation $\operatorname{E}_Q[\cdot]$ can be interchanged. We now show that this is possible given our assumptions. We have to distinguish two cases.

Interest rate is zero or negative We assumed cash flows to be nonnegative. Using Beppo Levi's dominated convergence theorem the result follows.

Interest rate is positive In this case the cash flow process is bounded from below by the real number K. Now, consider the transformed cash flow process $CF_t^* := CF_t + K$ that is nonnegative using our assumption. Also, the limes

$$\lim_{T\to\infty} \sum_{s=1}^{T} e^{-sr} \operatorname{E}_Q[CF_{s+t}^*|\mathcal{F}_t] = \frac{K}{e^r - 1} + \lim_{T\to\infty} \sum_{s=1}^{T} e^{-sr} \operatorname{E}_Q[CF_{s+t}|\mathcal{F}_t]$$

is finite. Using Beppo Levi's dominated convergence theorem we have

$$\lim_{T\to\infty} \sum_{s=1}^{T} e^{-sr} \operatorname{E}_Q[CF_{s+t}^*|\mathcal{F}_t] = \operatorname{E}_Q\left[\lim_{T\to\infty} \sum_{s=1}^{T} e^{-sr} CF_{s+t}^*|\mathcal{F}_t\right]$$

which then (by subtraction of $\frac{K}{e^r-1}$) implies that limes and expectation can be interchanged.

This finishes the proof. $\square$

Our new idea is the assumption of a stochastic discount rate. The first observation that we must take into account is that the cost of capital and dividend-price ratio will not necessarily be the same variables as in Gordon-Shapiro. Therefore, in the following we will no longer talk in terms of cost of capital, but in terms of the dividend-price ratio. This dividend-price ratio will be denoted by κ_t. The process κ_t has the following characteristics:

1. The ratio κ_t is positive and independent of cash flows CF_t.

2. The (inverse) ratio forms a Q-martingale, i.e.,

$$\mathrm{E}_Q[\kappa_{t+1}^{-1}|\mathcal{F}_t] = \kappa_t^{-1}. \tag{8}$$

   This requirement generalizes the assumption of constant cost of capital as in the traditional Gordon Shapiro model.

3. Finally, and this is crucial, the cash flows form an autoregressive process under Q with a stochastic coefficient, i.e.,

$$\mathrm{E}_Q[CF_{t+1}|\mathcal{F}_t] = \frac{e^r}{1+\kappa_t} CF_t. \tag{9}$$

Given these assumptions, the following holds true.

Proposition 2. *If the above assumptions for the dividend-price ratio κ apply, then the firm value is unique and given by* (3).

Proof. First we show that the cash flows satisfy (Cauchy). This then proves that the company value is unique and the first part of the theorem is proven. For this we first prove an inequality

$$\begin{aligned}
e^{-2r}\left|\mathrm{E}_Q[CF_{t+2}|\mathcal{F}_t]\right| &= e^{-r}\left|\mathrm{E}_Q\left[e^{-r}\,\mathrm{E}_Q[CF_{t+2}|\mathcal{F}_{t+1}]|\mathcal{F}_t\right]\right| \\
&= \left|\mathrm{E}_Q\left[\frac{e^{-r}}{1+\kappa_{t+1}} CF_{t+1}|\mathcal{F}_t\right]\right| && \text{see (9)} \\
&= \mathrm{E}_Q\left[\frac{\kappa_{t+1}^{-1}}{1+\kappa_{t+1}^{-1}}|\mathcal{F}_t\right]\ \left|e^{-r}\,\mathrm{E}_Q\left[CF_{t+1}|\mathcal{F}_t\right]\right| && \text{independence} \\
&\le \frac{\mathrm{E}_Q[\kappa_{t+1}^{-1}|\mathcal{F}_t]}{1+\mathrm{E}_Q[\kappa_{t+1}^{-1}|\mathcal{F}_t]}\ \frac{\kappa_t^{-1}}{1+\kappa_t^{-1}}|CF_t| && \text{Jensens inequ.} \\
&= \left(\frac{\kappa_t^{-1}}{1+\kappa_t^{-1}}\right)^2 |CF_t| && \text{see (8), (9)}
\end{aligned}$$

and therefore by induction

$$e^{-sr}\,|\mathrm{E}_Q[CF_{t+s}|\mathcal{F}_t]| \leq \left(\frac{\kappa_t^{-1}}{1+\kappa_t^{-1}}\right)^s |CF_t|.$$

Using this inequality we now verify the Cauchy criterion

$$\sup_{T_2 \geq T_1}\left|\sum_{s=T_1}^{T_2} e^{(t-s)r}\,\mathrm{E}_Q[CF_s|\mathcal{F}_t]\right| \leq |CF_t| \sup_{T_2\geq T_1}\sum_{s=T_1}^{T_2}\left(\frac{1}{1+\kappa_t}\right)^{s-t}.$$

This sequence converges for $T_2 \to \infty$ because $\kappa_t > 0$.

Since the value of the company is unique we now show that only the equation (3) is appropriate. We already know that the fundamental theorem (Fund) must hold if the value is given by (Value) (see proposition 1). If we now verify that $V_t = CF_t \kappa_t^{-1}$ satisfies the fundamental theorem the proof is finished. This can be established as follows:

$$\begin{aligned} e^{-r}\,\mathrm{E}_Q[CF_{t+1} + V_{t+1}|\mathcal{F}_t] &= V_t \\ e^{-r}\,\mathrm{E}_Q[CF_{t+1}|\mathcal{F}_t]\;\mathrm{E}_Q[1+\kappa_{t+1}^{-1}|\mathcal{F}_t] &= CF_t\,\kappa_t^{-1} \\ e^{-r}\,\mathrm{E}_Q[CF_{t+1}|\mathcal{F}_t] &= CF_t\,\frac{\kappa_t^{-1}}{1+\mathrm{E}_Q[\kappa_{t+1}^{-1}|\mathcal{F}_t]} \end{aligned}$$

Using $\mathrm{E}_Q[\kappa_{t+1}^{-1}|\mathcal{F}_t] = \kappa_t^{-1}$ our claim follows. □

We continue with the assumption that the cash flows are AR(1) under the probability P, see (4). In addition to the main result, we can show other characteristics. The condition $\mathrm{E}_Q[\kappa_{t+1}^{-1}|\mathcal{F}_t] = \kappa_t^{-1}$ implies that because of the discrete Girsanov theorem a $\mathcal{F}_{t-1}$-measurable process A_{t-1} exists such that:[11]

$$\mathrm{E}[\kappa_{t+1}^{-1}|\mathcal{F}_t] = \kappa_t^{-1} + A_{t-1}. \tag{10}$$

Then, using the subjective probability P the dividend-price ratio also appears in the definition of a cost of capital or return if $A_{t-1} \neq -1$,

$$\frac{\mathrm{E}[CF_{t+1} + V_{t+1}|\mathcal{F}_t]}{V_t} - 1 = e^g\,(1 + A_{t-1})\,\kappa_t.$$

[11] A_{t-1} is called the predictable covariation, see Föllmer and Schied (2004, Theorem 10.25).

2.4 Lucas model

As a final step, we will examine whether our approach can be reconciled with equilibrium considerations. To this end we take a consumption based model of Lucas (1978) as an orientation. This model describes the smoothing of consumption for an investor who has eternal life and is characterized by a certain "impatience". The utility of this investor is given by

$$\sum_{t=1}^{\infty} \beta^t \, \mathrm{E}\left[u(c_t)\right],$$

where $\beta < 1$ is the utility discount factor and c_t consumption at time t. Consumption is financed by the company we are looking at.

Using the usual Euler equations[12] and assuming our stochastic version of the Gordon-Shapiro formula results in an equation

$$1 = \beta \, \mathrm{E}\left[\frac{1+\kappa_{t+1}^{-1}}{\kappa_t^{-1}} \frac{CF_{t+1} u'(CF_{t+1})}{CF_t u'(CF_t)} | \mathcal{F}_t\right].$$

But since κ_{t+1} and CF_{t+1} are independent such an equality cannot hold unless consumption cancels completely. Even looking at the case $u(c) = \ln(c)$ results in an equation that cannot hold jointly with (10) as straightforward calculations show. Hence we must conclude that our stochastic Gordon-Shapiro formula does not to be compatible with a Lucas equilibrium.

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[12] See Ljungqvist and Sargent (2018, equation (1.3.3)).

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3 Appendix

In order to study AR(2) processes from a general point of view, one must use the theory of difference functions.[13] This is a rather complicated issue. For this reason, we will limit ourselves to a specific example in the following. Let the cash flows satisfy an equation of the form

$$\forall t \geq 0 \quad CF_{t+1} = e^{g'} CF_t + e^a CF_{t-1} + \varepsilon_{t+1},$$

where $g', a < r$ are deterministic. The random terms may be iid with $\mathrm{E}_Q[\varepsilon_t] = 0$. The cash flow $CF_{-1} = 0$ is exogenously predetermined and $CF_0 > 0$ is deterministic. Following the same procedure as for obtaining the valuation equation (6) we get

$$V_t = \frac{CF_t}{e^{r-g'} - 1} + \frac{CF_{t-1}}{e^{r-a} - 1}. \tag{11}$$

This allows us to examine the coefficients of dispersion of cash flows and firm values. Starting with the first cash flow results in

$$\begin{aligned} CF_1 &= e^{g'} CF_0 + \varepsilon_1 \\ \mathrm{E}[CF_1] &= e^{g'} CF_0 + \underbrace{\mathrm{E}[\varepsilon_1]}_{=:\mu} \\ \sigma[CF_1] &= \underbrace{\sigma[\varepsilon_1]}_{=:\sigma} \end{aligned}$$

For the next point in time one obtains

$$\begin{aligned} CF_2 &= e^{g'} CF_1 + e^a CF_0 + \varepsilon_2 \\ &= \left(e^{2g'} + e^a\right) CF_0 + e^{g'} \varepsilon_1 + \varepsilon_2 \\ \mathrm{E}[CF_2] &= \left(e^{2g'} + e^a\right) CF_0 + (1 + e^{g'})\mu \\ \sigma[CF_2] &= \sqrt{e^{2g'} + 1}\,\sigma. \end{aligned}$$

The coefficient of dispersion of cash flows is thus shown to be time-dependent. In summary, we can state that

$$\frac{\sigma[CF_1]}{\mathrm{E}[CF_1]} = \frac{\sigma}{e^{g'} CF_0 + \mu}, \quad \frac{\sigma[CF_2]}{\mathrm{E}[CF_2]} = \frac{\sqrt{e^{2g'} + 1}\,\sigma}{(e^{2g'} + e^a)\, CF_0 + (1 + e^{g'})\mu}$$

[13] See for instance Elaydi (2005, section 2.3).

applies. Looking at the firm value we get

$$\begin{aligned} \mathrm{E}[V_1] &= \frac{\mathrm{E}[CF_1]}{e^{g'} - e^r} + \frac{\mathrm{E}[CF_0]}{e^a - e^r} \\ &= \frac{e^{g'} CF_0 + \mu}{e^{g'} - e^r} + \frac{CF_0}{e^a - e^r} \end{aligned}$$

and

$$\begin{aligned} \sigma[V_1] &= \frac{\sigma[CF_1]}{e^{g'} - e^r} \\ &= \frac{\sigma}{e^{g'} - e^r} \end{aligned}$$

This results in

$$\frac{\sigma[V_1]}{\mathrm{E}[V_1]} = \frac{\sigma}{e^{g'} CF_0 + \mu + \frac{e^{g'} - e^r}{e^a - e^r} CF_0} .$$

It follows without further ado that the dispersion of V_1 is *smaller* than the dispersion of CF_1. And that is exactly what we wanted to show.

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