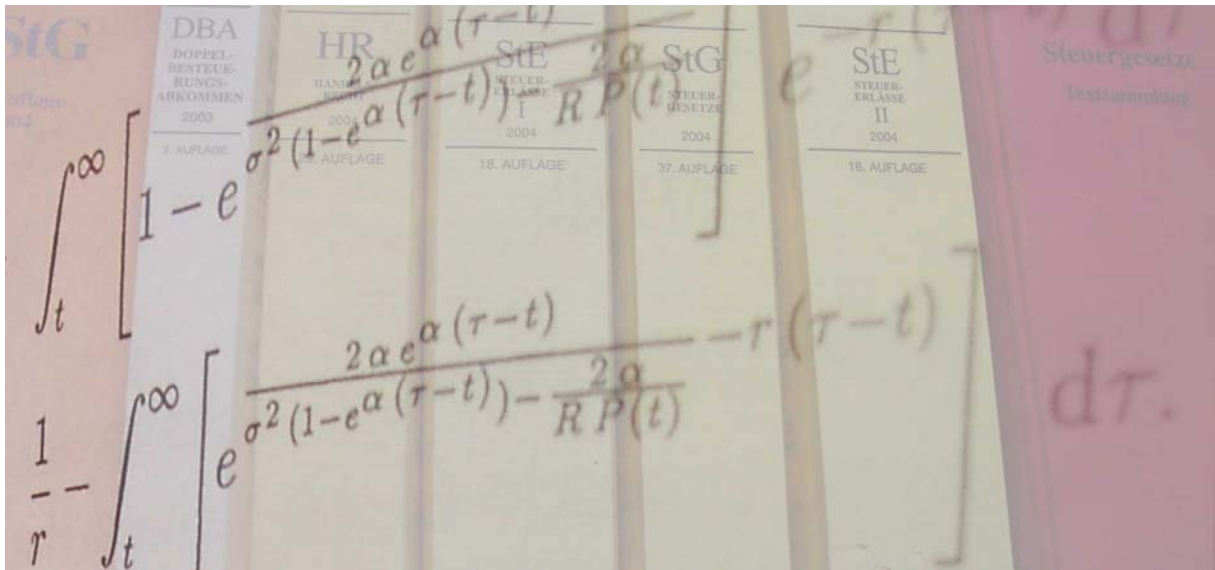


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Taxation of Risky Investment and Paradoxical Investor Behavior

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Abstract

Analysis of investment decisions under uncertainty does not provide a general analytical description of investor reactions towards profit tax rate changes. We use a real option model and find distorting tax treatment of risk-free and risky investment. We analytically identify general paradoxical settings and furthermore, a whole set of neutral tax rates (tax regimes) in case of tax rate changes. Unlike for other tax paradoxa neither depreciation rules nor loss offset restrictions are responsible for the paradoxical reaction. The implied ambiguity of tax effects under uncertainty, affects individual project evaluation and understanding of tax effects on aggregate.

Keywords: investment decisions, real options, tax paradox, uncertainty

JEL classification: H25; H21

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1 Introduction

The influence of taxes on investment decisions has been analyzed by public economics for many years. So-called neutral tax systems that do not affect investment decisions are often considered desirable from a tax policy perspective. Neutral tax systems may serve as a benchmark for identifying normal and paradoxical effects of tax changes on investment decisions and thus are helpful for individual tax planning activities and tax policy discussions. Deterministic examples of neutral tax systems are the cash flow tax (Brown 1948) and the taxation of true economic profit (Samuelson 1964 and Johansson 1969).

Economists have been especially interested in tax effects under uncertainty. Conditions for a neutral business taxation under uncertainty have been addressed by Bond and Devereux (1995). Under uncertainty and irreversibility, real option-based models (Dixit/Pindyck 1994; Trigeorgis 1996) are widely accepted for assessing investment projects. Enriching the real option literature by integrating taxation (e.g., Harchaoui/Lasserre 1996; Jou 2000; Pennings 2000; Agliardi 2001; Panteghini 2001, 2004; Niemann and Sureth 2004, Schneider, Dirk 2005) leads to investment rules that consider managerial flexibility, irreversibility and tax effects. Further, under specific assumptions it is possible to identify tax systems that are neutral with respect to investment decisions. For risk neutral investors, neutral tax systems have already been proved in the real option context by Niemann (1999). First results for neutral taxation under risk aversion have been presented by Niemann and Sureth (2004, 2005). As the discussion on tax systems and tax reforms is an on-going process (Auerbach and Hines 1988; Kaplow 1986, p. 607; Hammond 1990, p. 26) it is important to understand the effects of tax rate changes on investment decisions as well as distortions which might occur. So far the existing post-tax analyses do not provide a general analytical description of investor reactions to profit tax rate changes.

There are several theoretical and empirical studies examining the economic impact of taxation on risky investment decisions. Domar and Musgrave (1944) and later Schneider, Dieter (1980) and Konrad (1991) investigate the influence of proportional income taxes on risk-taking depending on loss offsetting rules. E.g., Stiglitz (1969) investigates the effects of capital gains taxes on the demand for risky assets.

Furthermore, there is a body of empirical papers on investor reactions to tax rate changes and tax reforms. Alvarez, Kanninen and Södersten (1998) investigate whether or not tax policy uncertainty is harmful for investment.¹ Lang and Shackelford (2000) empirically document the extent to which stock

¹Problems created by anticipated tax reforms have been addressed by Alvarez, Kanninen and Södersten (1998) as well. These questions go back to King (1974) and later Auerbach and Hines (1988), Robson (1989), Auerbach and Hassett (1992). In the following, we abstract from such anticipatory and transitional problems.

prices react to cuts in the capital gains tax rate. Shackelford and Verrecchia (2002) and Blouin, Raedy and Shackelford (2003) show that capital gains taxes lead investors to defer selling appreciated stock. Keuschnigg and Nielsen (2004) empirically analyze the influence of capital gains tax on start-up finance with double moral hazard. Corresponding to the findings of Poterba (1989a, 1989b), they point out that capital gains tax particularly discourages entrepreneurial efforts. Blouin, Hail, and Yetman (2005), Cook (2006) and Dai, Maydew, Shackelford, and Zhang (2006) examine empirically the response of equity values to the announcement of a decrease in the capital gains tax rate. Edmiston (2004) estimates tax volatility in a cross-country investigation and provides a panel regression suggesting that the volatility of effective tax rates on capital income has a significant negative impact on investment.

MacKie-Mason (1990) models nonlinear tax effects under uncertainty and demonstrates that policy may subsidize or discourage individual investment depending on the tax system. Altug, Demers and Demers (2001) examine the implications of tax risk and persistence on irreversible investment decisions theoretically. Panteghini and Scarpa (2003) show that regulatory risk may or may not affect negatively investment decisions. Gamba, Sick and León (2005) analyze the effect of uncertainty and debt financing on the real option value of an investment. Pawlina and Kort (2005) find that policy changes under uncertainty may have a non-monotonous impact on the investment threshold. Bloom, Bond and Van Reenen (2007) point out that companies' responsiveness to any given policy is much lower in periods of high uncertainty.

Beyond the identification of neutral tax systems, the existing real option-oriented analyses that take account of tax effects are rather limited and do not provide a general analytical description of so-called normal and paradoxical investor reactions to profit tax rate changes in this context. Either they fail to focus on this issue or they are limited to numerical investigations (e.g., Pawlina and Kort 2005, p. 1204).

Besides the well-known tax paradoxa under certainty caused either by depreciation allowances that exceed economic depreciation in present value terms (see Samuelson 1964) or by loss carry forwards, minimum taxation or wealth taxation (see, e.g., Auerbach and Poterba 1987, p. 319, 336; Niemann 2004, Kiesewetter and Niemann 2004, and Sureth and Maiterth 2005), we provide an analytical approach to identify tax paradoxa under uncertainty even by looking at nothing more than the uncertain stream of cash flows.

We implement a simple tax system and focus on investors that face risky irreversible investment opportunities which comprise an option to wait. We apply the Dixit-Pindyck (1994) paradigm. The investor compares the costs and benefits of investing immediately. If the investor observes a sufficiently high realization of the cash flow process, the project will be carried out. Taxation may cause distortions as

taxes asymmetrically affect risk-free and risky investment. We determine under which circumstances a marginal tax rate change discriminates or rather subsidizes a risky project in comparison to a risk-free alternative or even leaves the investment decision unaffected. Finally, we identify analytically general paradoxical settings and furthermore, describe tax rates for investment projects with specific characteristics (growth rate, market rate and volatility) that preserve the critical post-tax investment threshold in case of deterministic tax rate changes. We determine a whole set of neutral tax rates describing tax regimes under which specific risky investments are not distorted when tax rates change and enables us to distinguish between normal and paradoxical investment reactions.

Identifying normal and paradoxical tax regimes can be regarded as a first step to a generalized description of tax effects under uncertainty. The results are useful for tax rate discussions as they help to forecast the impact of tax rate changes on investment activities of specific types of investment projects. This is interesting information for a tax planning individual investor as well as for discussing the economic impact of tax reforms. From the viewpoint of an investor, investors can anticipate whether a risky project is discriminated, subsidized or treated neutrally by taxation if they know the type of tax regime for each investment project that complies with the required condition. Hence, facing tax rate changes tax planning will be easier, i.e. it is easier for an investor to forecast the tax effects. Furthermore, from the viewpoint of the government, it will be easy to identify the direction of distortion of tax rate changes and to control for tax policy effects at least for some types of investment project.

The remainder of this paper begins with a description of the model and a brief deduction of the critical investment threshold in section 2. In section 3 we introduce neutral tax regimes and distinguish analytically between normal, neutral and paradoxical tax effects in section 4. We summarize and draw some conclusions in section 5.

2 The model

General setting: In this partial analytic framework we analyze a risky investment opportunity including an option to invest. The investor may either realize the investment project and earn stochastic cash flow or postpone the investment, holding the option to invest while sacrificing cash flows and thereby avoiding unexpectedly low cash flows. The initial investment cost I_0 is given and constant. Cash flow uncertainty is summarized in an exogenously given single continuous-time stochastic process, P , following a geometric Brownian motion

$$\frac{dP}{P} = \alpha dt + \sigma dz \quad (1)$$

with a constant drift α and a constant volatility σ , where $\alpha, \sigma > 0$ and dz denotes the increment of a standard Wiener process.

Further, we assume the investment to be irreversible once it is accomplished, which implies that it is impossible to abandon a project during its economic life ending at time T . T is supposed to be infinite. Thus, the return from the project is given by the expected cash flow. The project's cash flow π is a function of the stochastic process P and time t : $\pi \equiv \pi(P, t)$. To simplify we set the pre-tax cash flow $\pi(P, t)$ equal to the geometric Brownian motion P : $\pi(P, t) = P(t, \alpha, \sigma)$.

There are two approaches to derive the optimal investment rule under uncertainty and to assess the value of the option to invest: dynamic programming and contingent claims analysis. Without taxes both approaches are extensively discussed in real option theory.² However, even considering that taxes have already been included in these analyses, the discussion is far from complete.³

In this model we would like to focus on effects arising from irreversibility and flexibility only, so we concentrate on the case of an investment into risky non-depreciable investment projects like listed or non-listed shares or land. We therefore exclude periodical tax-deductible depreciation allowances from our analysis.⁴ Hence, an investor faces the opportunity to invest in a risky project or alternatively a risk-free bond. Furthermore, we will assume a simple tax system with a proportional profit tax only. The investor's income consists of the post-tax cash flow from the risky investment that is a dividend payout. Taxable capital gains may not arise, as the investment is assumed to be irreversible and $T \rightarrow \infty$.

The tax base equals the cash flow $\pi = P$. The tax rate τ is assumed to be deterministic. The post-tax cash flow P_τ is defined as:

$$P_\tau = (1 - \tau) P. \quad (2)$$

If the investor does not realize the investment project funds may alternatively be invested into bonds and yield the risk-free capital market rate r that is assumed to be constant. The debit and credit rates are identical and the risk-free after-tax interest rate r_τ can be written as $r_\tau = (1 - \tau) r$. As the underlying risk-free financial investment is just a special case of a real investment project, whose return always equals true economic profit and herewith implies a neutral depreciation of zero, it may serve as yardstick.

²See Dixit and Pindyck (1994).

³For a post-tax comparison of the two approaches see Niemann/Sureth (2002).

⁴Concerning distortions caused by depreciation allowances see Sureth (1999, pp. 278-287) who identifies tax paradoxes caused by non-neutral depreciation allowances in a real option model with contingent claims analysis assuming a setting with temporary suspension and operating costs.

Investment decisions and critical threshold: In order to derive a rule for optimal investment, we have to determine the value of the underlying risky asset, the investment project. Once the project is realized, i.e. the investment object is acquired, the project does not involve any flexibility, and its economic value consists solely of its future cash flows. For a risk neutral investor the post-tax project value V_τ is given by its expected present value computed with the after-tax cash flow from the project P_τ and the risk-free after-tax market rate of return r_τ .

$$V_\tau \equiv V_\tau(P) = E \left[\int_s^\infty [(1 - \tau) P(t)] e^{-r_\tau(t-s)} dt \right]. \quad (3)$$

which finally is:

$$V_\tau(P) = \frac{(1 - \tau) P}{r_\tau - \alpha}; \quad r_\tau > \alpha. \quad (4)$$

Given the value of the underlying asset (4), the post-tax value of the option to invest F_τ can be determined applying dynamic programming. The investor wants to maximize

$$\max \{V_\tau^T - I_0, F_\tau^T\}$$

with

$$F_\tau(V_\tau) = \max \left\{ \max_T E \left[(V_\tau^T - I_0) e^{-r_\tau(T-t)} \right], 0 \right\}.$$

thus he will compare at every point in time the difference of the expected present value of the risky project and the initial outlay with the option value. The investor will give up the option to invest at an optimal time T and realize the project as soon as this difference is at least identical to the option value. Focussing on a non-depreciable option to invest we can determine the post-tax option value F_τ which requires the continuous-time Hamilton-Jacobi-Bellman equation⁵

$$r_\tau F_\tau dt \stackrel{!}{=} E[dF_\tau].$$

Applying Itô's lemma to the stochastic differential dF_τ we have to use the well-known boundary conditions

$$F_\tau(0) = 0, \quad (5)$$

$$F_\tau(P_\tau^*) = V_\tau(P_\tau^*) - I_0, \quad (6)$$

$$\frac{dF_\tau(P_\tau^*)}{dP} = \frac{dV_\tau(P_\tau^*)}{dP}. \quad (7)$$

Equation (5) implies that a call on a worthless underlying is itself worthless. The free boundary conditions equations (6) and (7) determine the transition from the continuation region to the exercise region at the

⁵Cf. Niemann/Sureth (2005), p. 81. For the pre-tax model cf. Dixit/Pindyck (1994), p. 141.

critical investment threshold P_τ^* . The so-called value-matching condition (6) ensures that the benefit from the project is equal to its costs at the point of transition. Equation (7) is called smooth-pasting condition requiring identity of marginal benefits and marginal costs at the critical threshold. Finally we obtain the value of the option

$$F_\tau(P) = A_\tau P^{\lambda_\tau}, \quad A_\tau > 0, \quad \lambda_\tau = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{\left(\frac{\alpha}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r_\tau}{\sigma^2}} > 1, \quad (8)$$

where A_τ is a constant factor to be determined. Solving for P_τ^* leads to the post-tax critical investment threshold for an investment into a risky project:⁶

$$P_\tau^* = \frac{\lambda_\tau}{\lambda_\tau - 1} \frac{r_\tau - \alpha}{1 - \tau} I_0 \text{ with } \lambda_\tau = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{\left(\frac{\alpha}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r_\tau}{\sigma^2}} > 1. \quad (9)$$

P_τ^* indicates whether or not the investment should be postponed. If the actually observed realization P_τ is higher than the critical value P_τ^* , the investment should be carried out immediately, otherwise it should be delayed until P_τ^* is reached.⁷

3 Distortion-free tax rate changes and neutral tax regimes

Since neutral tax systems are well-known under certainty and have already been derived under risk neutrality in real option literature,⁸ we will not discuss their properties in detail.

Here we look at investment rules for risky investment projects (e.g. investments in stocks on the capital market or other non-depreciable investment objects) compared to risk-free investments (e.g. bonds) when tax rates change. The investment decision depends on the expected growth rate α of cash flows generated by the risky project and the inherent volatility of cash flows captured by σ as well as the rate of return of the alternative risk-free investment r and the investor's individual tax rate τ . For all potential combinations of α , r , σ and τ we identify those tax rates where a change in τ does not affect the threshold ($\partial P_\tau^* / \partial \tau = 0$). In other words, for certain settings of α , r , σ we determine the tax rates which do not generate a distortion of the investment decision if tax rates change. Moreover, given certain conditions of the growth rate α , the interest rate r and σ we can state whether a deterministic change in the tax rate will foster future investment, make it less likely that an investment project will be realized or even leave the investment decision unchanged (neutral tax rate). As this tax rate is neutral only for a

⁶See appendix 1.

⁷To illustrate the impact of taxes on the threshold it is interesting to have a look at the pre-tax threshold which is: $P^* = \frac{\lambda}{\lambda-1} (r - \alpha) I_0$. Cf. Dixit and Pindyck (1994), p. 143.

⁸E.g. Bond and Devereux (1995); Panteghini (2001); Sureth (2002); Niemann and Sureth (2004).

specific investment project with the attributes given by the required combination of values for α , r , σ , we refer to such a tax rate as a (parameter-specific) neutral tax rate τ^N .

Having determined the critical investment thresholds, it is possible to derive parameter-specific neutral tax rates as just described above. On this basis, we can identify a whole set of neutral tax rates that we will refer to as *neutral tax regime* in the following. Such a neutral tax regime describes scenarios under which risky investments are not distorted when tax rates change. Given the environment with the parameters α , r , σ and assuming $I_0 = 1$ a tax regime can be described by these coefficients and a tax rate τ .

Definition 1 A *tax regime* is a set of points $(\alpha, r, \tau, \sigma) \subsetneq \mathbb{R}^4$. A tax regime is called **neutral** if $\frac{\partial P_\tau^*}{\partial \tau} = 0 \forall \tau \in [0, 1)$ with⁹

$$\frac{\partial P_\tau^*}{\partial \tau} = -\frac{\partial \lambda_\tau}{\partial \tau} \frac{r_\tau - \alpha}{1 - \tau} \frac{1}{(\lambda_\tau - 1)^2} - \frac{\lambda_\tau}{\lambda_\tau - 1} \frac{\alpha}{(1 - \tau)^2} = 0. \quad (10)$$

I.e., a marginal tax rate change has no effect on the critical threshold if the tax rate belongs to the neutral tax regime.

After having defined a neutral tax regime we would now like to look at the major properties of this tax regime. In other words, identifying a neutral regime enables us to describe the conditions for risky investment projects not suffering from distortions caused by tax rate changes. We show that there is a set of points that solves for the above condition $\frac{\partial P_\tau^*}{\partial \tau} = 0$. In order to capture all neutral combinations of α , r , τ and σ we first show that the *neutral tax regime* is a three-dimensional manifold. Second, we use the implicit function theorem to define neutral tax rates τ^N as a function of (α, r, σ) . $\tau^N = \tau^N(\alpha, r, \sigma)$ covers all possible neutral tax rates for variations in α , σ and r and thereby describes different possible neutral settings of various risky investment projects.

Proposition 1: *Let the cash flow of our investment project with cash flow P follow a geometric Brownian motion (1) and let the profit be taxed at the tax rate τ . Then the neutral tax regime (set of points $(\alpha, r, \tau, \sigma)$ with $\frac{\partial P_\tau^*}{\partial \tau} = 0$) with the growth rate α , the volatility σ and the risk-free market rate r forms a three-dimensional submanifold of the \mathbb{R}^4 .*

Using proposition 1,¹⁰ we can show that there is an implicit function for neutral tax rates τ^N which

⁹See appendix 2.

¹⁰For a proof of proposition 1 see appendix 3.

depends on α , r and σ . τ^N defines neutral tax rates, i.e. all tax rates, which do not change the investment decision for a marginal change in τ given a set of α , r and σ .

Proposition 2: *For each vector $(\alpha_0, r_0, \tau_0, \sigma_0)$ that fulfills condition (10) there is a marginal environment around this vector, such that τ^N is an implicit function of α , r and σ .*

$$\tau^N = \tau^N(\alpha, r, \sigma). \quad (11)$$

With the implicit function (11) we are able to describe neutral tax rates for the possible parameter settings.¹¹

4 Normal and paradox tax regimes

As shown in the previous section it is possible to identify settings for risky financial investments, where changes in the tax rate would not distort the investment decision. These settings may serve as a reference point. They enable a distinction to be made between settings with distortions and those without distortions if tax rates change.

As we are able to describe a neutral tax regime under uncertainty, it must be possible to identify tax regimes that are non-neutral. Thus, we can determine under which conditions a marginal tax rate change discriminates or rather subsidizes a risky project in comparison to a risk-free alternative or even leaves the investment decision unaffected. Among these regimes there will be tax regimes invoking a "normal" influence of taxation on the investment decision, i.e. an increase in the tax rate will lead to an increase in the critical threshold and thus to a postponement of the underlying investment. An investor who wants to invest immediately would then prefer to continue waiting and realize the risk-free investment instead. Furthermore, there will be other tax regimes invoking a paradoxical effect on the investment decision. I.e., an investor who integrates taxes in his decision calculus will be more likely to realize the risky project for a higher tax rate. Identifying normal and paradoxical tax regimes can be regarded as a first step to a general description of tax effects under uncertainty.

Neutral tax regime Before we turn to other than neutral regimes we would like to have a closer look at the characteristics of the neutral tax regime. With the help of the implicit function $\tau^N = \tau^N(\alpha, r, \sigma)$ we can discuss the shape and location of the manifold in different dimensions by looking at the relevant partial derivatives. As it is not possible to identify conditions for neutral tax regimes that hold for all

¹¹For a proof of proposition 2 see appendix 4.

possible parameter settings analytically we focus on settings with a sufficiently small difference between r_τ and α , i.e.

$$\varepsilon = r_\tau - \alpha \quad (12)$$

is small.¹² For sufficiently small ε we are able to distinguish exactly between the different types of tax regimes.

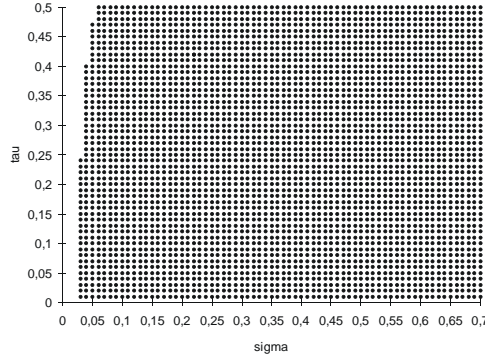


Figure 1: Sufficiently small ε

Numerical examples like e.g. the parameter combination $r = 0.05, \alpha = 0.02, \sigma = 0.25$ and $\tau = 0.35$ lead to a sufficiently small ε . Figure 1 illustrates a selection of σ and τ combinations that fulfill this condition for given $r = 0.05$ and $\alpha = 0.02$. These examples represent feasible combinations of parameters that allow us to identify a neutral tax regime. The example suggests that for relatively high volatilities, many typically observable combinations of α and r fulfill this condition.

Up to now effects of tax changes on investment decision under uncertainty have been mostly discussed as numerical examples. In this analysis we try to obtain general analytical results. If we assume small ε this will enable us to identify unambiguous normal, neutral and even paradoxical effects under more general conditions. Restricting the analysis to small ε does not mean that these effects do not exist for larger ε . It just means that we do not have general conditions for these regimes.

If ε in condition (12) is sufficiently small the signs of the partial derivatives of τ^N with respect to α , r and σ will be unambiguous for each project-specific setting.¹³

$$\frac{\partial \tau^N}{\partial \alpha} < 0, \quad \frac{\partial \tau^N}{\partial \sigma} < 0, \quad \frac{\partial \tau^N}{\partial r} > 0 \quad (13)$$

¹²An analytical description of sufficiently small ε is given in appendix 4.

¹³See appendix 4.

Normal and paradox tax regimes With the help of the neutral tax regime we can distinguish between regions with normal reactions of the critical threshold and paradoxical reactions when tax rates rise.

Definition 2 A tax reaction is called normal if an increase in the tax rate increases the required threshold $P_\tau^*, \frac{\partial P_\tau^*}{\partial \tau} > 0$.

Definition 3 A tax reaction is called paradox if an increase in the tax rate decreases the required threshold $P_\tau^*, \frac{\partial P_\tau^*}{\partial \tau} < 0$.

Proposition 3: If tax rates are higher/lower than the rates of the neutral tax regime, the reactions of the threshold are normal/paradox.

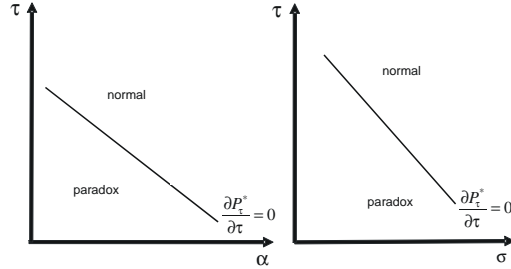


Figure 2: Neutral tax regime and partial derivatives

Figure 2 illustrates the linearized partial shape of the function for the neutral tax rate τ^N and the growth rate α and risk σ . We can identify the corresponding regions for normal and paradoxical tax effects. The graphs indicate the location of the different regimes in each dimension. The described different regimes reflect the general effect of taxes on investment decisions under uncertainty depending on the characteristics of the underlying investment project given by α and r and σ . If c.p. former deterministic cash flows become stochastic and an investor faces an option to invest rising tax rates may be neutral for the investment decision, or may even switch the sign of the reaction under certainty.

What is the economics of switching the sign of the threshold caused by uncertainty? Under uncertainty the option to invest has an own economic value. Hence the net present value of an investment which is the objective value of the investor includes this component. Consequently, the present value of the option

affects the decision. However, taxes affect the benefit from waiting (value of holding period) differently than they do the other components of the investment decision. The contribution of the expected cash flow from the investment and of the option to the net present value of the whole project are treated asymmetrically by taxation. We observe two major effects:

One arises from the tax treatment of the option and thereby is induced directly by uncertainty. As the increase in the option value during the holding period is not subject to tax and a corresponding economic appreciation for tax purposes is missing, the option enjoys a tax privilege.¹⁴

The second effect is caused by taxation in a continuous-time growth model and thus is an effect that can be identified under certainty as well. At time t all realized cash flows are subject to tax. In contrast, the growth of cash flows that will be realized during the infinitesimal small period t will become tax-liable at $t+dt$. Consequently, this marginal return and growth will be temporarily tax-exempt invoking asymmetric treatment of the underlying riskless and risky investments. Under certainty (perfect foresight) an investor and the public sector would know about this marginal return and would be able to burden it with taxes. Under uncertainty both agents have no more than expectations about this marginal return. Thus, exact taxation is not possible. This effect until now has not been treated and analyzed in the literature.

This effect from continuous-time modelling may exert a different influence on the threshold than the one from the tax privilege of the option. We can show that, depending on the type of tax regime, the direct effect from uncertainty may be stronger or weaker than the reaction from the after-tax growth process and hence, the effect from the component addressing uncertainty may or may not overcompensate the second effect and overall change the sign of the reaction of the threshold.

Looking at the reaction in figure 2 we see the following mechanics from taxation and option pricing. Assume an investment project with a given growth rate α is just taxed at a neutral tax rate τ^N . Neutral tax rates τ^N are drawn as a decreasing function of α and σ . Now, we assume the tax rate to rise and future cash flows from the investment to be taxed at this higher rate τ with $\tau > \tau^N$. As the option is part of the value of the opportunity to invest in a risky project and further the option is tax-favored a rise in τ implies an increase of this tax privilege. The relative advantage from holding the option grows. Consequently, an investor will be willing to abandon the option and carry out the risky project only for relative higher values of P . Thus, increasing the tax rate will increase the critical investment threshold which is a normal reaction. The relative advantage from holding the option increases and the critical

¹⁴Cf. Niemann (1999).

threshold will be higher. Exploring the environmental conditions under which the original threshold would be preserved in case of a tax rate rise, we discover that a simultaneous decrease in α that leads to a decrease in P_τ^* as described above may compensate for the tax rate effect. Then, under the resulting new setting with decreasing α and given r and σ we would fall back to a neutral regime (negative slope of the τ^N -curve).

The reaction below the τ^N -function is quite different. Again, with rising tax rates when $\tau < \tau^N$ the component of the threshold covering the option value increases. C.p. this effect from the option pushes up the critical threshold. However, in the paradox regime we realize that the tax-benefit from the option is now overcompensated by an opposing effect. This second effect arises from the temporal tax-exemption of (continuous) growth in the present tax period. Whereas a realized cash flow from either the risky investment project or the risk-less investment into bonds is cut proportionally by the tax rate τ , the investor's benefit from simultaneous growth of revenues (α) during each period is tax-exempt as it does not become an instantaneously realized cash flow during the same period. Therefore, an asymmetric effect of taxes favours the risky investment project. C.p from this asymmetry we obtain a partial decrease in the threshold when the tax rate increases. In the paradox regime this second effect is overcompensating the first effect. Hence, the higher the tax rate the more attractive becomes the risky project. If $\tau < \tau^N$ this effect from asymmetric taxation of projects with continuously growing cash flow and an investment into a bond overcompensates the tax impact on the option values arising from uncertainty. Overall, the increase in the tax rate causes a reduction of the investment threshold. The investor faces a paradox situation.

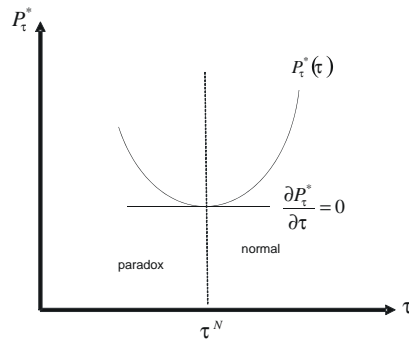


Figure 3: Tax rate variations and tax regimes

These reactions for the different regimes are also depicted in figure 3. For given external conditions the

reaction of the threshold to an increase in the tax rate is described. To the right of τ^N increasing taxes will cause the expected increase in the threshold (*normal reaction*). The increase in the threshold may lead to reject the project that was favorable before. To the left of τ^N increasing taxes will decrease the threshold and improve the evaluation of the uncertain investment project (*paradoxical reaction*).

In figure 4 we draw the shape and location of the neutral tax regime applying a three-dimensional illustration of the neutral tax rates depending on α and σ . The plane separates the different regimes and shows that even in the underlying simple case of an investment in a risky financial project, i.e. in a non-depreciable project, uncertainty may change the sign of the investment reaction on tax rate changes.

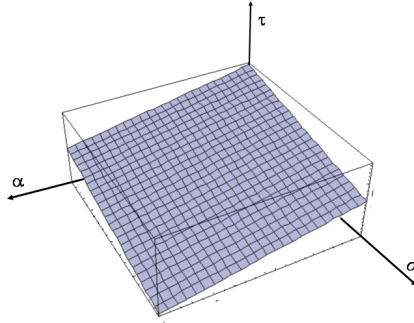


Figure 4: Neutral tax regime

This figure illustrates the relation between α , σ and τ . Obviously, neutral tax rates need to decrease with increasing growth rates, whereas they need to rise to compensate for decreasing risk.

We see in figure 4 that an increasing risk (rising σ) will increase the value of the waiting time. Being able to wait and not having to start immediately has an increasing economic value. The relatively high value of the waiting time is pushing up the threshold, as the investor wants to be compensated for higher risk. If we are looking for tax rates that preserve the threshold in case of higher risk there needs to be a compensation in the tax parameters for bearing more risk. This compensation can be achieved by a decrease in the tax τ . A simultaneously decreasing tax rate would adjust the threshold and lead back to the neutral tax condition.

The discussion on a change in the growth rate α is similar. If α increases this will push up the critical threshold as holding the option implies rather rapid growth and thus a high value of the waiting time. An investor will therefore ask for a relatively high compensation if he gives up the option to invest. As

the increase in the value of the cash flow during waiting time is not tax-liable, high tax rates amplify the option's benefit. Hence, if α increases lower tax rates are necessary to provide neutrality in the above defined sense.

Having identified the two asymmetries when taxing risky investment projects it seems beneficial to introduce a tax system under certainty that treats projects with continuously growing cash flows and bonds in the same way. This would require a depreciation term that is a function of α , r and σ eliminating this asymmetry. If we implement additionally an ex-post adjustment mechanism to balance out the deviation between expected cash flow and realized cash flows, then just the tax effect from the option would occur under uncertainty. Under these conditions we would observe exclusively the tax effect from the option and hence the investor will always face a normal reaction if $r_\tau > \alpha$ holds. However, the adjustment procedure dissolves continuous-time modelling.

Furthermore, to neutralize all the asymmetries from taxation either an economic appreciation or depreciation (corresponding to economic depreciation known from taxing true economic profit) could be introduced into tax law. As such an adjustment rule is not included in the underlying and in real-world tax systems, an asymmetry remains. Speaking more generally, it seems beneficial to introduce a tax system that treats projects with continuously growing cash flows, bonds and options in the same way. This would require a adjustment term that is a function of α , r and σ eliminating this asymmetry.

Our analytical results can be confirmed by numerical examples. In these examples we can see normal, neutral and paradoxical scenarios. Moreover, we find some - at first glance - surprising results like the example given in figure 5. Whenever the standard deviation is rather small paradoxical tax effects occur. However as can be seen in the right graph in figure 5 we do not just obtain one single neutral tax rate, several local minima are found each accompanied by normal and paradoxical regions. Obviously, multiple tax paradoxa occur if the volatility is low or if the difference between the growth rate α and r_τ is sufficiently small. However, in most cases we observe a normal reaction of the investment threshold P_τ^* towards tax rate changes.

What does this imply? If investors decide on high risk and high growth projects, rising tax rate will increase the critical value. For investment projects with low risk or generally high similarity to the risk free asset normal, neutral and even paradoxical reactions can be expected. Due to the high non linearity, and the high number of different normal and paradox tax rate intervals, then tax effects become rather complex and ambiguous. Tax reformers face problems when they try to estimate the effects of tax policies.

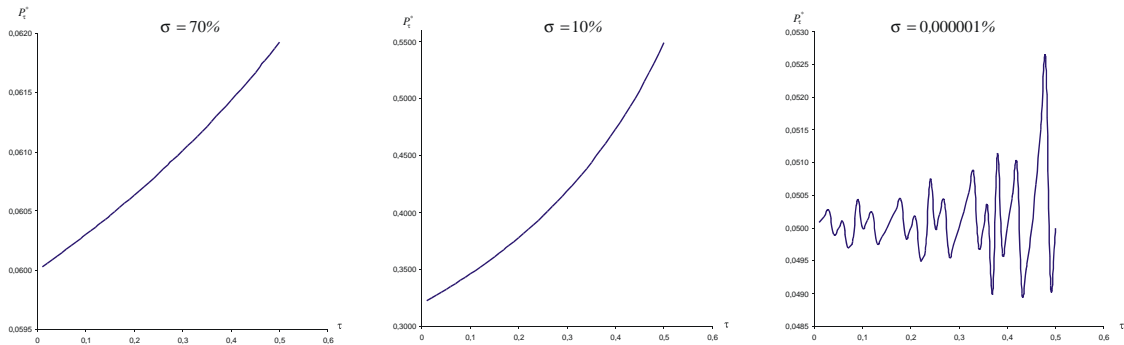


Figure 5: Numerical normal and paradoxical tax effects on the critical threshold for $r = 5\%$ and $\alpha = 2\%$

5 Conclusions

The effect of tax rate changes on investments will change substantially if uncertainty and irreversibility is included in the investment decision. Using a real option model with dynamic programming for risky non-depreciable irreversible investments, a simple tax system with profit tax only and a cash flow that follows a geometric Brownian motion, we can identify three regimes of tax effects on investment decisions. In contrast to the existing literature that usually falls back on numerical analyses we succeed in identifying analytically sets of tax rates for which an increase in tax rates will lead to the expected increase in the threshold and hence a decrease in investments. Our findings are general whenever the differential between the growth rate and the market rate of return is sufficiently small. This set of tax rates is called a normal tax regime. There is also a set of tax rates, where an increase in tax rates will not cause any effects on the threshold and hence investment decision. This set of tax rates is referred to as a parameter-specific neutral tax regime. However, there is a set of tax rates where an increase in tax rates will even decrease the threshold and favour the risky investment. These unexpected reactions are called paradox. Unlike for other tax paradoxa neither depreciation rules¹⁵ nor loss offset restrictions are responsible for the observed paradoxical reaction.

What is the economics of these paradoxical reactions? Under uncertainty the option to invest has a positive economic value. Taxes affect the benefit from waiting (value of holding period) differently than they do the other components of the investment decision. The contribution of the expected cash flow from the investment and of the option to the net present value of the whole project are treated asymmetrically by taxation. Whereas realized cash flow from the risk-free investment into bonds is cut proportionally by the tax rate, the investor's benefit from the option during the holding period is completely tax-exempt

¹⁵In fact there is a depreciation rule that is equal to zero. This rule is not identical to economic depreciation.

as it does not become a realized cash flow. The non realized increase in stochastic cash flows during the potential period of waiting is not taxed. Furthermore, the marginal return and growth from the risky project is temporarily tax-exempt.

Identifying these regimes is interesting from two perspectives: From the viewpoint of an investor, investors can anticipate whether a risky project is discriminated, subsidized or treated neutrally by tax rate changes knowing the type of tax regime. From the viewpoint of the government, it will be easier to identify the direction of distortion of tax rate reforms. Further, as the analysis is looking at a single project with its environment and the environment is described by the growth rate and volatility of the cash flow as well as the return of the risk-free investment, tax rate changes may have opposite effects on the different investment projects. Depending on the external conditions the same change in tax rates may have normal, neutral as well as paradoxical effects on different projects. Numerical analyses clarify that tax planning for investors as well as for tax reformers becomes impossible if volatility is low.

To ensure a desired influence of taxation on investment behavior tax reformers need to know whether the majority of investment projects is situated rather in the paradox or in the normal region (i.e., right or left of τ^N). The tax effect on aggregate investment becomes generally ambiguous.

Extended Appendix

Annotation 1: Properties of λ_τ :

$$\begin{aligned}
 \lambda_\tau &> 1 \quad \text{because} \quad Q(1) < 0 \quad \text{see } (??) \\
 \text{we assume small } \varepsilon \text{ with } \varepsilon &= r_\tau - \alpha = \left(\frac{r_\tau}{\alpha} - 1\right) \alpha > 0 \\
 \text{hence } \lambda_\tau &= \frac{1}{2} - \frac{r_\tau - \varepsilon}{\sigma^2} + \left[\left(\frac{1}{2} - \frac{r_\tau - \varepsilon}{\sigma^2} \right)^2 + \frac{2r_\tau}{\sigma^2} \right]^{\frac{1}{2}} \\
 \text{and further } \lim_{\varepsilon \rightarrow 0} \lambda_\tau &= 1, \quad \lim_{\sigma \rightarrow \infty} \lambda_\tau = 1. \quad \text{see } (??)
 \end{aligned}$$

Corollary 1: The assumption $r_\tau > \alpha$ is equivalent to

$$\lambda_\tau - \frac{r_\tau}{\alpha} < 0 \quad \text{see } (??)$$

Proof of corollary 1:

$$\begin{aligned}
 \text{It applies } \frac{r_\tau}{\alpha} &< \left(\frac{r_\tau}{\alpha}\right)^2 \quad \text{as } r_\tau > \alpha \\
 \Leftrightarrow \left(\frac{1}{2} - \frac{\alpha}{\sigma^2}\right)^2 + \frac{2r_\tau}{\sigma^2} &< \left(\frac{1}{2} - \frac{\alpha}{\sigma^2}\right)^2 - \frac{r_\tau}{\alpha} + \frac{2r_\tau}{\sigma^2} + \left(\frac{r_\tau}{\alpha}\right)^2 \\
 \Leftrightarrow \left(\frac{1}{2} - \frac{\alpha}{\sigma^2}\right)^2 + \frac{2r_\tau}{\sigma^2} &< \left(\frac{1}{2} - \frac{\alpha}{\sigma^2}\right)^2 - \frac{2r_\tau}{\alpha} \left(\frac{1}{2} - \frac{\alpha}{\sigma^2}\right) + \left(\frac{r_\tau}{\alpha}\right)^2 \\
 \Leftrightarrow \left(\frac{1}{2} - \frac{\alpha}{\sigma^2}\right)^2 + \frac{2r_\tau}{\sigma^2} &< \left[-\left(\frac{1}{2} - \frac{\alpha}{\sigma^2}\right) + \frac{r_\tau}{\alpha} \right]^2 \\
 \Leftrightarrow \left[\left(\frac{1}{2} - \frac{\alpha}{\sigma^2}\right)^2 + \frac{2r_\tau}{\sigma^2} \right]^{\frac{1}{2}} &< -\left(\frac{1}{2} - \frac{\alpha}{\sigma^2}\right) + \frac{r_\tau}{\alpha} \\
 \Leftrightarrow \overbrace{\left[\left(\frac{1}{2} - \frac{\alpha}{\sigma^2}\right)^2 + \frac{2r_\tau}{\sigma^2} \right]^{\frac{1}{2}}}^{\lambda_\tau} + \frac{1}{2} - \frac{\alpha}{\sigma^2} &< \frac{r_\tau}{\alpha} \\
 \Leftrightarrow \lambda_\tau - \frac{r_\tau}{\alpha} &< 0
 \end{aligned}$$

Corollary 2: $0 < \delta < 1$ and finite

Proof of corollary 2:

$$\begin{aligned}\lambda_\tau - \frac{r_\tau}{\alpha} &< 0 \Leftrightarrow \lambda_\tau - 1 < \frac{r_\tau}{\alpha} - 1 \\ 0 &< 1 - \frac{\lambda_\tau - 1}{\frac{r_\tau}{\alpha} - 1} \\ \delta &\equiv 1 - \frac{\lambda_\tau - 1}{\frac{r_\tau}{\alpha} - 1} > 0\end{aligned}$$

$$\begin{aligned}\frac{\lambda_\tau - 1}{\frac{r_\tau}{\alpha} - 1} + \delta &= \frac{(\lambda_\tau - 1)\alpha}{\varepsilon} + \delta = 1 \Leftrightarrow 1 - \delta = \frac{(\lambda_\tau - 1)\alpha}{\varepsilon} \quad \text{see (??)} \\ \lim_{\varepsilon \rightarrow 0} \frac{(\lambda_\tau - 1)\alpha}{\varepsilon} &: \quad \text{using L'Hopital's rule:} \\ \lim_{\varepsilon \rightarrow 0} \frac{\alpha \frac{\partial \lambda_\tau}{\partial \varepsilon}}{1} &= \frac{\alpha \left[\left(\frac{1}{2} - \frac{r_\tau - \varepsilon}{\sigma^2} \right)^2 + \frac{2r_\tau}{\sigma^2} \right]^{-\frac{1}{2}}}{\sigma^2} \lambda_\tau > 0 \text{ and finite} \\ \text{hence} &: \quad 1 - \delta \text{ is finite}\end{aligned}$$

Derivatives for λ_τ :

$$\begin{aligned}\frac{\partial \lambda_\tau}{\partial \tau} &= - \left[\left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right)^2 + \frac{2r(1-\tau)}{\sigma^2} \right]^{-1/2} \frac{r}{\sigma^2} < 0 \\ \frac{\partial^2 \lambda_\tau}{\partial \tau^2} &= - \left[\left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right)^2 + \frac{2(r-r\tau)}{\sigma^2} \right]^{-1/2-1} \left(\frac{r}{\sigma^2} \right)^2 < 0 \\ \frac{\partial \lambda_\tau}{\partial \alpha} &= -\frac{1}{\sigma^2} - \frac{2}{2} \left[\left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right)^2 + \frac{2(r-r\tau)}{\sigma^2} \right]^{\frac{1}{2}-1} \left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right) \frac{1}{\sigma^2} \\ &= -\frac{1}{\sigma^2} \left[1 + \left[\left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right)^2 + \frac{2(r-r\tau)}{\sigma^2} \right]^{-\frac{1}{2}} \left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right) \right] < 0 \\ &= -\frac{\left[\left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right)^2 + \frac{2(r-r\tau)}{\sigma^2} \right]^{-\frac{1}{2}}}{\sigma^2} \left[\left[\left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right)^2 + \frac{2(r-r\tau)}{\sigma^2} \right]^{\frac{1}{2}} + \left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right) \right] < 0 \\ &= -\frac{\left[\left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right)^2 + \frac{2(r-r\tau)}{\sigma^2} \right]^{-\frac{1}{2}}}{\sigma^2} \lambda_\tau < 0 \\ \frac{\partial \lambda_\tau}{\partial \varepsilon} &= \frac{1}{\sigma^2} + \frac{2}{2} \left[\left(\frac{1}{2} - \frac{r_\tau - \varepsilon}{\sigma^2} \right)^2 + \frac{2r_\tau}{\sigma^2} \right]^{-\frac{1}{2}} \left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right) \frac{1}{\sigma^2} \\ &= \frac{\left[\left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right)^2 + \frac{2(r-r\tau)}{\sigma^2} \right]^{-\frac{1}{2}}}{\sigma^2} \lambda_\tau\end{aligned}$$

$$\frac{\partial \lambda_\tau}{\partial r} = \left[\left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right)^2 + \frac{2r(1-\tau)}{\sigma^2} \right]^{-\frac{1}{2}} \frac{(1-\tau)}{\sigma^2} > 0$$

$$\begin{aligned} \frac{\partial \lambda_\tau}{\partial \sigma} &= \frac{2\alpha}{\sigma^3} + \frac{1}{2} \left(\left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right)^2 + \frac{2r_\tau}{\sigma^2} \right)^{-\frac{1}{2}} \left(2 \left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right) \cdot \frac{2\alpha}{\sigma^3} - \frac{4r_\tau}{\sigma^3} \right) < 0 \\ &= \frac{2\alpha}{\sigma^3} + \left(\left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right)^2 + \frac{2r_\tau}{\sigma^2} \right)^{-\frac{1}{2}} \left(\left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right) \cdot \frac{2\alpha}{\sigma^3} - \frac{2r_\tau}{\sigma^3} \right) < 0 \\ &= \frac{2\alpha}{\sigma^3} \left[1 + \left(\left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right)^2 + \frac{2r_\tau}{\sigma^2} \right)^{-\frac{1}{2}} \left(\frac{1}{2} - \frac{\alpha}{\sigma^2} - \frac{r_\tau}{\alpha} \right) \right] < 0 \\ &= \frac{2\alpha \left(\left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right)^2 + \frac{2r_\tau}{\sigma^2} \right)^{-\frac{1}{2}}}{\sigma^3} \left[\left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right)^2 + \frac{2r_\tau}{\sigma^2} \right]^{\frac{1}{2}} + \frac{1}{2} - \frac{\alpha}{\sigma^2} - \frac{r_\tau}{\alpha} < 0 \end{aligned}$$

Annotation 2: Properties of the investment threshold Derivatives of P_τ^* with respect to the tax

rate: $\frac{\partial P_\tau^*}{\partial \tau}$

$$P_\tau^* = \frac{\lambda_\tau}{\lambda_\tau - 1} \frac{r_\tau - \alpha}{1 - \tau} I_0 \quad (14)$$

for $I_0 = 1$ we find

$$\frac{\partial P_\tau^*}{\partial \tau} = -\frac{\partial \lambda_\tau}{\partial \tau} \frac{r_\tau - \alpha}{1 - \tau} \frac{1}{(\lambda_\tau - 1)^2} - \frac{\lambda_\tau}{\lambda_\tau - 1} \frac{\alpha}{(1 - \tau)^2} \quad (15)$$

Lemma 1: Let $G = \frac{\partial P_\tau^*}{\partial \tau} = -\frac{\partial \lambda_\tau}{\partial \tau} \frac{r_\tau - \alpha}{1 - \tau} \frac{1}{(\lambda_\tau - 1)^2} - \frac{\lambda_\tau}{\lambda_\tau - 1} \frac{\alpha}{(1 - \tau)^2}$ and $\varepsilon \equiv r_\tau - \alpha > 0$ then

$$\frac{\partial^2 P_\tau^*}{\partial \tau^2} = \frac{dG}{d\tau} > 0 \quad \text{for sufficiently small } \varepsilon$$

Proof of Lemma 1: Since $G = \frac{\partial P_\tau^*}{\partial \tau} = -\frac{\partial \lambda_\tau}{\partial \tau} \frac{r_\tau - \alpha}{1 - \tau} \frac{1}{(\lambda_\tau - 1)^2} - \frac{\lambda_\tau}{\lambda_\tau - 1} \frac{\alpha}{(1 - \tau)^2}$ and $\varepsilon \equiv r_\tau - \alpha > 0$ then

$$\begin{aligned} \frac{\partial^2 P_\tau^*}{\partial \tau^2} &= \frac{dG}{d\tau} = -\frac{\partial^2 \lambda_\tau}{\partial \tau^2} \frac{r_\tau - \alpha}{1 - \tau} \frac{1}{(\lambda_\tau - 1)^2} + \frac{\partial \lambda_\tau}{\partial \tau} \frac{r}{1 - \tau} \frac{1}{(\lambda_\tau - 1)^2} - \frac{\partial \lambda_\tau}{\partial \tau} \frac{r_\tau - \alpha}{(1 - \tau)^2} \frac{1}{(\lambda_\tau - 1)^2} \\ &\quad + \frac{\partial \lambda_\tau}{\partial \tau} \frac{r(1 - \tau) - \alpha}{1 - \tau} \frac{2}{(\lambda_\tau - 1)^3} \frac{\partial \lambda_\tau}{\partial \tau} - \frac{\frac{\partial \lambda_\tau}{\partial \tau}}{\lambda_\tau - 1} \frac{\alpha}{(1 - \tau)^2} \\ &\quad + \frac{\lambda_\tau \frac{\partial \lambda_\tau}{\partial \tau}}{(\lambda_\tau - 1)^2} \frac{\alpha}{(1 - \tau)^2} - \frac{\lambda_\tau}{\lambda_\tau - 1} \frac{\alpha 2(1 - \tau)}{(1 - \tau)^4} \\ &= -\frac{\partial^2 \lambda_\tau}{\partial \tau^2} \frac{r_\tau - \alpha}{1 - \tau} \frac{1}{(\lambda_\tau - 1)^2} + \frac{\partial \lambda_\tau}{\partial \tau} \frac{r_\tau}{(1 - \tau)^2} \frac{1}{(\lambda_\tau - 1)^2} - \frac{\partial \lambda_\tau}{\partial \tau} \frac{r_\tau - \alpha}{(1 - \tau)^2} \frac{1}{(\lambda_\tau - 1)^2} \\ &\quad + \frac{\partial \lambda_\tau}{\partial \tau} \frac{r(1 - \tau) - \alpha}{1 - \tau} \frac{2}{(\lambda_\tau - 1)^3} \frac{\partial \lambda_\tau}{\partial \tau} - \frac{\frac{\partial \lambda_\tau}{\partial \tau}}{(\lambda_\tau - 1)^2} \frac{(\lambda_\tau - 1)\alpha}{(1 - \tau)^2} \\ &\quad + \frac{\lambda_\tau \frac{\partial \lambda_\tau}{\partial \tau}}{(\lambda_\tau - 1)^2} \frac{\alpha}{(1 - \tau)^2} - \frac{\lambda_\tau}{\lambda_\tau - 1} \frac{\alpha 2(1 - \tau)}{(1 - \tau)^4} \end{aligned}$$

$$\begin{aligned}
&= -\frac{\partial^2 \lambda_\tau}{\partial \tau^2} \frac{r_\tau - \alpha}{1 - \tau} \frac{1}{(\lambda_\tau - 1)^2} + \frac{\partial \lambda_\tau}{\partial \tau} \frac{r_\tau}{(1 - \tau)^2} \frac{1}{(\lambda_\tau - 1)^2} - \frac{\partial \lambda_\tau}{\partial \tau} \frac{r_\tau - \alpha}{(1 - \tau)^2} \frac{1}{(\lambda_\tau - 1)^2} \\
&\quad + \frac{\partial \lambda_\tau}{\partial \tau} \frac{r(1 - \tau) - \alpha}{1 - \tau} \frac{2}{(\lambda_\tau - 1)^3} \frac{\partial \lambda_\tau}{\partial \tau} + \frac{\partial \lambda_\tau}{\partial \tau} \frac{1}{(\lambda_\tau - 1)^2} \frac{\alpha}{(1 - \tau)^2} - \frac{\lambda_\tau}{\lambda_\tau - 1} \frac{\alpha 2}{(1 - \tau)^3} \\
&= -\frac{\partial^2 \lambda_\tau}{\partial \tau^2} \frac{\varepsilon(1 - \tau)}{(1 - \tau)^2} \frac{1}{(\lambda_\tau - 1)^2} + \frac{\partial \lambda_\tau}{\partial \tau} \frac{\varepsilon(1 - \tau)}{(1 - \tau)^2} \frac{2}{(\lambda_\tau - 1)^3} \frac{\partial \lambda_\tau}{\partial \tau} \\
&\quad + 2 \frac{\partial \lambda_\tau}{\partial \tau} \frac{1}{(\lambda_\tau - 1)^2} \frac{\alpha}{(1 - \tau)^2} - \frac{\lambda_\tau(\lambda_\tau - 1)}{(\lambda_\tau - 1)^2} \frac{\alpha 2}{(1 - \tau)^3} \\
&= -\frac{\partial^2 \lambda_\tau}{\partial \tau^2} \frac{\varepsilon(1 - \tau)}{(1 - \tau)^2} \frac{1}{(\lambda_\tau - 1)^2} + 2 \frac{\varepsilon(1 - \tau)}{(1 - \tau)^2} \frac{1}{(\lambda_\tau - 1)^3} \left(\frac{\partial \lambda_\tau}{\partial \tau} \right)^2 \\
&\quad + \frac{\partial \lambda_\tau}{\partial \tau} \frac{2}{(\lambda_\tau - 1)^2} \frac{\alpha}{(1 - \tau)^2} - \frac{\lambda_\tau(\lambda_\tau - 1)}{(\lambda_\tau - 1)^2} \frac{\alpha 2}{(1 - \tau)^3} \\
&= \frac{1}{(\lambda_\tau - 1)^2} \frac{1}{(1 - \tau)^2} \left[\varepsilon(1 - \tau) \left[-\frac{\partial^2 \lambda_\tau}{\partial \tau^2} + \frac{2}{(\lambda_\tau - 1)} \left(\frac{\partial \lambda_\tau}{\partial \tau} \right)^2 \right] \right. \\
&\quad \left. + \frac{\partial \lambda_\tau}{\partial \tau} 2\alpha - \lambda_\tau(\lambda_\tau - 1) \frac{\alpha 2}{(1 - \tau)} \right]
\end{aligned}$$

As $\lambda_\tau = \frac{1}{2} - \frac{r_\tau - \varepsilon}{\sigma^2} + \left[\left(\frac{1}{2} - \frac{r_\tau - \varepsilon}{\sigma^2} \right)^2 + \frac{2r_\tau}{\sigma^2} \right]^{\frac{1}{2}}$ tends to 1 for sufficiently decreasing ε , the term $-\frac{(1 - \tau)\alpha(\lambda_\tau - 1)}{(1 - \delta)} \frac{\partial^2 \lambda_\tau}{\partial \tau^2} - \lambda_\tau(\lambda_\tau - 1) \frac{2\alpha}{(1 - \tau)}$ tends to become sufficiently close to zero. Further, as $0 < \delta < 1$ we must check the sign of $-\frac{(1 - \tau)\alpha(\lambda_\tau - 1)}{(1 - \delta)} \frac{2}{(\lambda_\tau - 1)} \left(\frac{\partial \lambda_\tau}{\partial \tau} \right)^2 - 2 \frac{\partial \lambda_\tau}{\partial \tau} \alpha$: for sufficiently small ε :

$$\begin{aligned}
\frac{\partial^2 P_\tau^*}{\partial \tau^2} &= \frac{dG}{d\tau} > 0 \quad \text{if} \quad \frac{(\lambda_\tau - 1)\alpha}{1 - \delta} \frac{2(1 - \tau)}{(\lambda_\tau - 1)} \left(\frac{\partial \lambda_\tau}{\partial \tau} \right)^2 + \frac{\partial \lambda_\tau}{\partial \tau} 2\alpha > 0 \\
2 \frac{\partial \lambda_\tau}{\partial \tau} \left[\frac{(1 - \tau)\alpha}{1 - \delta} \frac{\partial \lambda_\tau}{\partial \tau} + \alpha \right] &> 0
\end{aligned}$$

$$\begin{aligned}
0 &> -\frac{(1 - \tau)}{1 - \delta} \left[\left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right)^2 + \frac{2r(1 - \tau)}{\sigma^2} \right]^{-1/2} \frac{r}{\sigma^2} + 1 \\
0 &< \frac{r_\tau}{1 - \delta} \left[\left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right)^2 + \frac{2r(1 - \tau)}{\sigma^2} \right]^{-1/2} \frac{1}{\sigma^2} - 1
\end{aligned}$$

$$\begin{aligned}
1 &< \frac{r_\tau \varepsilon}{(\lambda_\tau - 1)\alpha\sigma^2 \left[\left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right)^2 + \frac{2r(1 - \tau)}{\sigma^2} \right]^{1/2}} \quad \text{using } \varepsilon = \frac{(\lambda_\tau - 1)\alpha}{1 - \delta} \\
A_1 &: = \frac{\varepsilon r_\tau}{(\lambda_\tau - 1)\alpha\sigma^2 \left[\left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right)^2 + \frac{2r(1 - \tau)}{\sigma^2} \right]^{1/2}} \quad \text{using L'Hopital's rule and } \varepsilon \rightarrow 0
\end{aligned}$$

$$\begin{aligned}
\lim_{\varepsilon \rightarrow 0} A_1 &= \frac{r_\tau}{\frac{\partial \lambda_\tau}{\partial \varepsilon} \alpha \sigma^2 \left[\left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right)^2 + \frac{2r(1-\tau)}{\sigma^2} \right]^{1/2}} \\
&= \frac{r_\tau}{\frac{\left[\left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right)^2 + \frac{2(r-r\tau)}{\sigma^2} \right]^{-\frac{1}{2}}}{\lambda_\tau \alpha \sigma^2} \left[\left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right)^2 + \frac{2r(1-\tau)}{\sigma^2} \right]^{1/2}} \\
&= \frac{r_\tau}{\lambda_\tau \alpha} \\
1 &< \frac{r_\tau}{\lambda_\tau \alpha} \Leftrightarrow \lambda_\tau < \frac{r_\tau}{\alpha} \quad \text{see (??)}
\end{aligned}$$

Hence, if ε decreases there will be a sufficiently small ε so that

$$\frac{\partial^2 P_\tau^*}{\partial \tau^2} = \frac{dG}{d\tau} > 0.$$

Numerical examples like e.g. the parameter combination $r = 0.05, \alpha = 0.02, \sigma = 0.25$ and $\tau = 0.35$ lead to a sufficiently small ε . This examples represents a feasible combination of parameters that allow us to identify a neutral tax regime.

Annotation 3: Neutral tax regime forms a manifold:

Proof of Proposition 1: Our investment threshold P_τ^* is given by

$$P_\tau^* = \frac{\lambda_\tau}{\lambda_\tau - 1} \frac{r_\tau - \alpha}{1 - \tau} \quad \text{see (9)}$$

for $I_0 = 1$.

For a neutral tax regime condition (10) must hold. I.e.

$$\frac{\partial P_\tau^*}{\partial \tau} = -\frac{\partial \lambda_\tau}{\partial \tau} \frac{r_\tau - \alpha}{1 - \tau} \frac{1}{(\lambda_\tau - 1)^2} - \frac{\lambda_\tau}{\lambda_\tau - 1} \frac{\alpha}{(1 - \tau)^2} = 0 \quad \text{see (10)}.$$

Now we will need the notion "regular value". A differentiable function f has the regular value y if for all $x \in f^{-1}(y)$ the derivative $Df(x)$ has a full rank.

As the derivative of G with respect to τ is $\frac{dG}{d\tau} = \frac{\partial^2 P_\tau^*}{\partial \tau^2} > 0$ (see Lemma 1, Annotation 3b), 0 is a regular value of $G : R^4 \rightarrow R$ and the set of points $G^{-1}(0)$ is a manifold of dimension $4 - 1 = 3$ (see Milnor, J.W., 1997. Topology from the differentiable viewpoint. Princeton University Press, Princeton, p. 11).

Annotation 4: Implicit function

Proof of Proposition 2: As $G^{-1}(0)$ is a manifold and as for each vector $(\alpha_0, r_0, \tau_0, \sigma_0)$ the derivative $\frac{dG}{d\tau}(\alpha_0, r_0, \tau_0, \sigma_0)$ is positive and as the partial derivatives of G by τ , α , r and σ are continuous, we can apply the implicit function theorem. Hence for a marginal environment of any vector $(\alpha_0, r_0, \tau_0, \sigma_0)$, τ^N is an implicit function of α , r and σ .
q.e.d.

Lemma 2: Let $\frac{\partial P_\tau^*}{\partial \tau} = G = -\frac{\partial \lambda_\tau}{\partial \tau} \frac{r_\tau - \alpha}{1 - \tau} \frac{1}{(\lambda_\tau - 1)^2} - \frac{\lambda_\tau}{\lambda_\tau - 1} \frac{\alpha}{(1 - \tau)^2} = 0$ and $\varepsilon \equiv r_\tau - \alpha > 0$ then

$$\text{a) } \frac{\partial G}{\partial \alpha} > 0, \quad \text{b) } \frac{\partial G}{\partial r} < 0 \quad \text{and} \quad \text{c) } \frac{\partial G}{\partial \sigma} > 0 \quad \text{for sufficiently small } \varepsilon$$

Proof of Lemma 2:

$$\begin{aligned} \text{a) } \frac{\partial G}{\partial \alpha} &= -\frac{\partial^2 \lambda_\tau}{\partial \alpha \partial \tau} \frac{r_\tau - \alpha}{1 - \tau} \frac{1}{(\lambda_\tau - 1)^2} + \frac{\partial \lambda_\tau}{\partial \tau} \frac{1}{1 - \tau} \frac{1}{(\lambda_\tau - 1)^2} \\ &\quad + \frac{\partial \lambda_\tau}{\partial \tau} \frac{r_\tau - \alpha}{1 - \tau} \frac{2 \frac{\partial \lambda_\tau}{\partial \alpha}}{(\lambda_\tau - 1)^3} + \frac{\frac{\partial \lambda_\tau}{\partial \alpha} \alpha}{(1 - \tau)^2 (1 - \tau)^2} - \frac{\lambda_\tau}{\lambda_\tau - 1} \frac{1}{(1 - \tau)^2} \\ &= \frac{1}{(\lambda_\tau - 1)^2 (1 - \tau)^2} \left[\begin{aligned} &-\frac{\partial^2 \lambda_\tau}{\partial \alpha \partial \tau} \varepsilon (1 - \tau) + \frac{\partial \lambda_\tau}{\partial \tau} (1 - \tau) \\ &+ \frac{\partial \lambda_\tau}{\partial \tau} \frac{2\varepsilon(1 - \tau) \frac{\partial \lambda_\tau}{\partial \alpha}}{(\lambda_\tau - 1)} + \frac{\frac{\partial \lambda_\tau}{\partial \alpha} \alpha - \lambda_\tau (\lambda_\tau - 1)}{\frac{\partial \lambda_\tau}{\partial \alpha}} \end{aligned} \right] > 0 \end{aligned}$$

Given $\varepsilon = \frac{(\lambda_\tau - 1)\alpha}{1 - \delta}$, we must show that $[...] < 0$:

$$0 < -\frac{\partial^2 \lambda_\tau}{\partial \alpha \partial \tau} \varepsilon (1 - \tau) + \frac{\partial \lambda_\tau}{\partial \tau} (1 - \tau) + \frac{\partial \lambda_\tau}{\partial \tau} \frac{2\varepsilon (1 - \tau) \frac{\partial \lambda_\tau}{\partial \alpha}}{(\lambda_\tau - 1)} + \frac{\frac{\partial \lambda_\tau}{\partial \alpha} \alpha - \lambda_\tau (\lambda_\tau - 1)}{\frac{\partial \lambda_\tau}{\partial \alpha}}$$

with ε sufficiently small, $(\lambda_\tau - 1)$ tends to become sufficiently close to zero.

Hence we obtain

$$\begin{aligned} 0 &< \frac{\frac{\partial \lambda_\tau}{\partial \tau} (1 - \tau)}{\frac{\partial \lambda_\tau}{\partial \tau} (1 - \tau)} + \frac{\frac{\partial \lambda_\tau}{\partial \tau} (\lambda_\tau - 1) \alpha}{\frac{\partial \lambda_\tau}{\partial \tau} (1 - \delta)} \frac{2(1 - \tau) \frac{\partial \lambda_\tau}{\partial \alpha}}{(\lambda_\tau - 1)} + \frac{\frac{\partial \lambda_\tau}{\partial \alpha} \alpha}{\frac{\partial \lambda_\tau}{\partial \alpha}} \\ 0 &< \frac{\partial \lambda_\tau}{\partial \tau} (1 - \tau) + \frac{\partial \lambda_\tau}{\partial \tau} \frac{2(1 - \tau) \alpha}{(1 - \delta)} \frac{\frac{\partial \lambda_\tau}{\partial \alpha}}{\frac{\partial \lambda_\tau}{\partial \alpha}} + \frac{\frac{\partial \lambda_\tau}{\partial \alpha} \alpha}{\frac{\partial \lambda_\tau}{\partial \alpha}} \end{aligned}$$

$$0 < \frac{\left[\left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right)^2 + \frac{2r_\tau}{\sigma^2} \right]^{-\frac{1}{2}}}{\sigma^2} \left[-r_\tau + 2 \frac{r_\tau \alpha}{(1 - \delta)} \frac{\left[\left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right)^2 + \frac{2r_\tau}{\sigma^2} \right]^{-\frac{1}{2}}}{\sigma^2} \lambda_\tau - \lambda_\tau \alpha \right]$$

$$\begin{aligned}
0 &< \left[-r_\tau + 2 \frac{r_\tau \alpha}{(1-\delta)} \frac{\left[\left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right)^2 + \frac{2r_\tau}{\sigma^2} \right]^{-\frac{1}{2}}}{\sigma^2} \lambda_\tau - \lambda_\tau \alpha \right] \\
0 &< -r_\tau + 2 \frac{r_\tau \alpha}{(1-\delta)} \frac{\left[\left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right)^2 + \frac{2r_\tau}{\sigma^2} \right]^{-\frac{1}{2}}}{\sigma^2} \lambda_\tau - \lambda_\tau \alpha \\
r_\tau + \lambda \alpha &< 2 \frac{r_\tau \alpha}{(1-\delta)} \frac{\left[\left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right)^2 + \frac{2r_\tau}{\sigma^2} \right]^{-\frac{1}{2}}}{\sigma^2} \lambda_\tau \\
1 &< 2 \frac{r_\tau \alpha}{(1-\delta)(r_\tau + \lambda_\tau \alpha)} \frac{\left[\left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right)^2 + \frac{2r_\tau}{\sigma^2} \right]^{-\frac{1}{2}}}{\sigma^2} \lambda_\tau \quad \text{with } \varepsilon = \frac{(\lambda_\tau - 1)\alpha}{1-\delta} \\
A_2 &: = \frac{2\varepsilon r_\tau}{(\lambda_\tau - 1)(r_\tau + \lambda_\tau(r_\tau - \varepsilon))} \frac{\lambda_\tau}{\sigma^2 \left[\left(\frac{1}{2} - \frac{(r_\tau - \varepsilon)}{\sigma^2} \right)^2 + \frac{2r_\tau}{\sigma^2} \right]^{\frac{1}{2}}} \quad \text{using L'Hopital's rule:} \\
\lim_{\varepsilon \rightarrow 0} A_2 &= \lim_{\varepsilon \rightarrow 0} \frac{2r_\tau \lambda_\tau}{(r_\tau + \lambda_\tau \alpha) \frac{\partial \lambda_\tau}{\partial \varepsilon} \sigma^2 \left[\left(\frac{1}{2} - \frac{(r_\tau - \varepsilon)}{\sigma^2} \right)^2 + \frac{2r_\tau}{\sigma^2} \right]^{\frac{1}{2}}} \\
&= \frac{2r_\tau \lambda_\tau}{(r_\tau + \lambda_\tau \alpha) \frac{\left[\left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right)^2 + \frac{2(r_\tau - r_\tau)}{\sigma^2} \right]^{-\frac{1}{2}}}{\sigma^2} \lambda_\tau \sigma^2 \left[\left(\frac{1}{2} - \frac{(r_\tau - \varepsilon)}{\sigma^2} \right)^2 + \frac{2r_\tau}{\sigma^2} \right]^{\frac{1}{2}}} \\
\lim_{\varepsilon \rightarrow 0} A_2 &= \frac{2r_\tau}{(r_\tau + \lambda_\tau \alpha)} \\
1 &< \frac{2r_\tau}{(r_\tau + \lambda_\tau \alpha)} \\
r_\tau + \lambda_\tau \alpha &< 2r_\tau \\
\frac{r_\tau}{\alpha} + \lambda_\tau &< 2 \frac{r_\tau}{\alpha} \iff \lambda_\tau < \frac{r_\tau}{\alpha}
\end{aligned}$$

$$\frac{\partial G}{\partial \alpha} > 0 \text{ for sufficiently small } \varepsilon.$$

$$\begin{aligned}
\text{b) } \frac{\partial G}{\partial r} &= -\frac{\partial^2 \lambda_\tau}{\partial r \partial \tau} \frac{r_\tau - \alpha}{1 - \tau} \frac{1}{(\lambda_\tau - 1)^2} - \frac{\partial \lambda_\tau}{\partial \tau} \frac{1}{(\lambda_\tau - 1)^2} \\
&\quad + \frac{(-)}{\partial \tau} \frac{r_\tau - \alpha}{1 - \tau} \frac{2 \frac{(+)}{\partial r} \lambda_\tau}{(\lambda_\tau - 1)^3} + \frac{(+)}{(1 - \tau)^2} \frac{\frac{\partial \lambda_\tau}{\partial r} \alpha}{(\lambda_\tau - 1)^2} \\
&\quad - \frac{\frac{\partial \lambda_\tau}{\partial r}}{(\lambda_\tau - 1)^2} \frac{(\lambda_\tau - 1) \alpha}{(1 - \tau)^2} + \frac{\lambda_\tau \frac{\partial \lambda_\tau}{\partial r}}{(\lambda_\tau - 1)^2} \frac{\alpha}{(1 - \tau)^2} \\
&= -\frac{\partial^2 \lambda_\tau}{\partial r \partial \tau} \frac{r_\tau - \alpha}{(1 - \tau)^2} \frac{(1 - \tau)}{(\lambda_\tau - 1)^2} - \frac{\partial \lambda_\tau}{\partial \tau} \frac{(1 - \tau)^2}{(1 - \tau)^2} \frac{1}{(\lambda_\tau - 1)^2} \\
&\quad + \frac{(-)}{\partial \tau} \frac{r_\tau - \alpha}{(1 - \tau)^2} \frac{2(1 - \tau) \frac{(+)}{\partial r} \lambda_\tau}{(\lambda_\tau - 1)^3} + \frac{(+)}{(1 - \tau)^2} \frac{\frac{\partial \lambda_\tau}{\partial r} \alpha}{(\lambda_\tau - 1)^2} \\
&= \frac{1}{(\lambda_\tau - 1)^2 (1 - \tau)^2} \left[-\frac{\partial^2 \lambda_\tau}{\partial r \partial \tau} \varepsilon (1 - \tau) - \frac{\partial \lambda_\tau}{\partial \tau} \frac{(1 - \tau)^2}{(\lambda_\tau - 1)} \right. \\
&\quad \left. + \frac{\partial \lambda_\tau}{\partial \tau} \frac{2\varepsilon(1 - \tau) \frac{(+)}{\partial r} \lambda_\tau}{(\lambda_\tau - 1)} + \frac{\partial \lambda_\tau}{\partial r} \alpha \right] < 0
\end{aligned}$$

$$\begin{aligned}
0 &> -\frac{\partial^2 \lambda_\tau}{\partial r \partial \tau} \varepsilon (1 - \tau) - \frac{\partial \lambda_\tau}{\partial \tau} (1 - \tau)^2 + \frac{\partial \lambda_\tau}{\partial \tau} \frac{2\varepsilon(1 - \tau) \frac{(+)}{\partial r} \lambda_\tau}{(\lambda_\tau - 1)} + \frac{\partial \lambda_\tau}{\partial r} \alpha \\
0 &> -\frac{\partial^2 \lambda_\tau}{\partial r \partial \tau} \varepsilon (1 - \tau) - \frac{\partial \lambda_\tau}{\partial \tau} (1 - \tau)^2 + \frac{\partial \lambda_\tau}{\partial \tau} \frac{(\lambda_\tau - 1) \alpha}{1 - \delta} \frac{2(1 - \tau) \frac{(+)}{\partial r} \lambda_\tau}{(\lambda_\tau - 1)} + \frac{\partial \lambda_\tau}{\partial r} \alpha \\
0 &> -\frac{\partial \lambda_\tau}{\partial \tau} (1 - \tau)^2 + \frac{\partial \lambda_\tau}{\partial \tau} \frac{2(1 - \tau) \alpha \frac{(+)}{\partial r} \lambda_\tau}{1 - \delta} + \frac{\partial \lambda_\tau}{\partial r} \alpha
\end{aligned}$$

$$\begin{aligned}
0 &> \left[\left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right)^2 + \frac{2r(1 - \tau)}{\sigma^2} \right]^{-1/2} \frac{r}{\sigma^2} (1 - \tau)^2 \\
&\quad - \left[\left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right)^2 + \frac{2r(1 - \tau)}{\sigma^2} \right]^{-1/2} \frac{r}{\sigma^2} \frac{\alpha(1 - \tau)}{1 - \delta} 2 \left[\left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right)^2 + \frac{2r(1 - \tau)}{\sigma^2} \right]^{-\frac{1}{2}} \frac{(1 - \tau)}{\sigma^2} \\
&\quad + \left[\left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right)^2 + \frac{2r(1 - \tau)}{\sigma^2} \right]^{-\frac{1}{2}} \frac{(1 - \tau)}{\sigma^2} \alpha
\end{aligned}$$

$$\begin{aligned}
0 &> \left[\left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right)^2 + \frac{2r(1-\tau)}{\sigma^2} \right]^{-1/2} \frac{1}{\sigma^2} \left[r_\tau(1-\tau) - \frac{2r_\tau\alpha(1-\tau)}{1-\delta} \left[\left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right)^2 + \frac{2r(1-\tau)}{\sigma^2} \right]^{-\frac{1}{2}} \frac{1}{\sigma^2} \right. \\
&\quad \left. + (1-\tau)\alpha \right] \\
0 &> r_\tau(1-\tau) - \frac{2r_\tau\alpha(1-\tau)}{1-\delta} \left[\left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right)^2 + \frac{2r(1-\tau)}{\sigma^2} \right]^{-\frac{1}{2}} \frac{1}{\sigma^2} + (1-\tau)\alpha \\
0 &< -r_\tau + \frac{2r_\tau\alpha}{1-\delta} \left[\left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right)^2 + \frac{2r(1-\tau)}{\sigma^2} \right]^{-\frac{1}{2}} \frac{1}{\sigma^2} - \alpha \\
(r_\tau + \alpha) &< + \frac{2\alpha r_\tau}{(1-\delta)\sigma^2} \left[\left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right)^2 + \frac{2r(1-\tau)}{\sigma^2} \right]^{-\frac{1}{2}} \\
1 &< \frac{2\alpha r_\tau}{(1-\delta)(r_\tau + \alpha)\sigma^2} \left[\left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right)^2 + \frac{2r(1-\tau)}{\sigma^2} \right]^{-\frac{1}{2}} \quad \text{with } \varepsilon = \frac{(\lambda_\tau - 1)\alpha}{1-\delta}
\end{aligned}$$

$$\begin{aligned}
A_3 &: = \frac{2\varepsilon r_\tau}{(\lambda_\tau - 1)(r_\tau + \alpha)\sigma^2 \left[\left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right)^2 + \frac{2r(1-\tau)}{\sigma^2} \right]^{\frac{1}{2}}} \quad \text{using L'Hopital's rule and } \varepsilon \rightarrow 0 \\
\lim_{\varepsilon \rightarrow 0} A_3 &= \lim_{\varepsilon \rightarrow 0} \frac{2r_\tau}{\frac{\partial \lambda_\tau}{\partial \varepsilon} (r_\tau + \alpha)\sigma^2 \left[\left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right)^2 + \frac{2r(1-\tau)}{\sigma^2} \right]^{\frac{1}{2}}} \\
&= \frac{2r_\tau}{\frac{\left[\left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right)^2 + \frac{2(r-r_\tau)}{\sigma^2} \right]^{-\frac{1}{2}}}{\sigma^2} \lambda (r_\tau + \alpha)\sigma^2 \left[\left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right)^2 + \frac{2r(1-\tau)}{\sigma^2} \right]^{\frac{1}{2}}}
\end{aligned}$$

$$\text{hence } 1 < \frac{2r_\tau}{\lambda(r_\tau + \alpha)} \quad \text{if } \lambda \text{ tends to } 1.$$

$$\frac{\partial G}{\partial r} > 0 \text{ for sufficiently small } \varepsilon.$$

$$\begin{aligned}
\text{c) } \frac{\partial G}{\partial \sigma} &= -\frac{\partial^2 \lambda_\tau}{\partial \sigma \partial \tau} \frac{r_\tau - \alpha}{1-\tau} \frac{1}{(\lambda_\tau - 1)^2} + \frac{\frac{(-)}{\partial \lambda_\tau} r_\tau - \alpha}{\partial \tau} \frac{1}{1-\tau} \frac{2 \frac{(-)}{\partial \lambda_\tau}}{(\lambda_\tau - 1)^3} + \frac{\frac{(-)}{\partial \lambda_\tau} \alpha}{(1-\tau)^2 (\lambda_\tau - 1)^2} > 0 \\
&= \frac{1}{(\lambda_\tau - 1)^2 (1-\tau)^2} \left[-\frac{\partial^2 \lambda_\tau}{\partial \sigma \partial \tau} (r_\tau - \alpha) (1-\tau) + \frac{\frac{(-)}{\partial \lambda_\tau} 2(r_\tau - \alpha)(1-\tau) \frac{(-)}{\partial \lambda_\tau}}{(\lambda_\tau - 1)} \right. \\
&\quad \left. + \frac{\frac{(-)}{\partial \lambda_\tau} \alpha}{\partial \sigma} \right] > 0
\end{aligned}$$

for sufficiently small ε , λ tends to one and hence $\frac{(-)}{\partial \lambda_\tau} \frac{\alpha}{1-\delta} \frac{2 \frac{(-)}{\partial \lambda_\tau}}{1-\tau}$ dominates.

Properties of the implicit function: For the implicit function: τ^N we can take the derivative with respect to α , $\frac{\partial \tau^N}{\partial \alpha}$, r , $\frac{\partial \tau^N}{\partial r}$ and σ , $\frac{\partial \tau^N}{\partial \sigma}$. Using the condition for a neutral tax regime (10) and assuming sufficiently small ε we obtain:

$$\begin{aligned} \frac{\partial \tau^N}{\partial \alpha} &= -\frac{\frac{\partial G}{\partial \alpha}}{\frac{\partial G}{\partial \tau}} \\ &= -\frac{-\frac{\partial^2 \lambda_\tau}{\partial \alpha \partial \tau} \varepsilon (1-\tau) + \frac{\partial \lambda_\tau}{\partial \tau} (1-\tau) + \frac{(-)}{\frac{\partial \lambda_\tau}{\partial \tau}} \frac{2\varepsilon(1-\tau) \frac{(-)}{\frac{\partial \lambda_\tau}{\partial \alpha}}}{(\lambda_\tau-1)} + \frac{(-)}{\frac{\partial \lambda_\tau}{\partial \alpha}} \alpha - \lambda_\tau (\lambda_\tau - 1)}{-\varepsilon (1-\tau) \left[\frac{\partial^2 \lambda_\tau}{\partial \tau^2} - \frac{2\left(\frac{\partial \lambda_\tau}{\partial \tau}\right)^2}{(\lambda_\tau-1)} \right] + 2\frac{\partial \lambda_\tau}{\partial \tau} \alpha - \lambda_\tau (\lambda_\tau - 1) \frac{\alpha^2}{(1-\tau)}} < 0 \end{aligned}$$

$$\begin{aligned} \frac{\partial \tau^N}{\partial r} &= -\frac{\frac{\partial G}{\partial r}}{\frac{\partial G}{\partial \tau}} \\ &= -\frac{\frac{(-)}{\frac{\partial^2 \lambda_\tau}{\partial r \partial \tau}} (r_\tau - \alpha) (1-\tau) - \frac{\partial \lambda_\tau}{\partial \tau} (1-\tau) + \frac{\partial \lambda_\tau}{\partial \tau} \frac{2(r_\tau - \alpha)(1-\tau) \frac{(+)}{\frac{\partial \lambda_\tau}{\partial r}}}{(\lambda_\tau-1)} + \frac{(+)}{\frac{\partial \lambda_\tau}{\partial r}} \alpha}{-\varepsilon (1-\tau) \left[\frac{\partial^2 \lambda_\tau}{\partial \tau^2} - \frac{2\left(\frac{\partial \lambda_\tau}{\partial \tau}\right)^2}{(\lambda_\tau-1)} \right] + 2\frac{\partial \lambda_\tau}{\partial \tau} \alpha - \lambda_\tau (\lambda_\tau - 1) \frac{\alpha^2}{(1-\tau)}} > 0 \end{aligned}$$

$$\begin{aligned} \frac{\partial \tau^N}{\partial \sigma} &= -\frac{\frac{\partial G}{\partial \sigma}}{\frac{\partial G}{\partial \tau}} \\ &= -\frac{\frac{(-)}{\frac{\partial^2 \lambda_\tau}{\partial \sigma \partial \tau}} (r_\tau - \alpha) (1-\tau) + \frac{(-)}{\frac{\partial \lambda_\tau}{\partial \tau}} \frac{2(r_\tau - \alpha)(1-\tau) \frac{(-)}{\frac{\partial \lambda_\tau}{\partial \sigma}}}{(\lambda_\tau-1)} + \frac{(-)}{\frac{\partial \lambda_\tau}{\partial \sigma}} \alpha (\lambda_\tau - 1)}{-\varepsilon (1-\tau) \left[\frac{\partial^2 \lambda_\tau}{\partial \tau^2} - \frac{2\left(\frac{\partial \lambda_\tau}{\partial \tau}\right)^2}{(\lambda_\tau-1)} \right] + 2\frac{\partial \lambda_\tau}{\partial \tau} \alpha - \lambda_\tau (\lambda_\tau - 1) \frac{\alpha^2}{(1-\tau)}} < 0 \end{aligned}$$

We therefore know that there is a marginal environment around τ^N where the described reaction can be observed.

Proof of Proposition 3: In a marginal environment of the neutral tax regime there is the function that defines the neutral tax rates $\tau^N = \tau^N(\alpha, r, \tau, \sigma)$. We know from lemma 1 that $\frac{\partial^2 P_\tau^*}{\partial \tau^2} > 0$, hence

$$\frac{\partial P_\tau^*}{\partial \tau} \left\{ \begin{array}{l} < \\ > \end{array} \right\} 0 \quad \text{for} \quad \tau \left\{ \begin{array}{l} < \\ > \end{array} \right\} \tau^N(\alpha, r, \tau, \sigma).$$

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